The Topology of Spaces of Polygons

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Abstract

We study the topology of spaces of polygons in Euclidean space, viewed up to translations. The main results concern the structure of the homology groups and of the cohomology rings of the spaces. In particular, it is shown that the spaces are classified by their $\mathbf{Z}_2$-cohomology rings. A principal tool used in the proofs is a new lacunary principle for Morse-Bott functions, which may be of independent interest. Several applications are discussed.

Declaration

This thesis is based on research carried out in the Pure Mathematics Group at the Department of Mathematical Sciences, Durham University. No part of this thesis has been submitted elsewhere for any other degree or qualification.

Some of the material is based on contributions to joint research with Professor Michael Farber. These results can be found in the articles [9] and [10] as well as in the preprint [11], submitted for publication. Most of the results of Chapter 3 and the material in the appendix are independent research and appear here for the first time.

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Contents

Chapter 0

Introduction

Spaces of polygons in Euclidean space $\mathbb{R}^d$ are particularly well-understood in the cases of dimensions $d = 2$ and $d = 3$. Most of the known results revolve around the configuration spaces defined by considering polygons up to the action of the full group of orientation-preserving Euclidean isometries. In the planar case $d = 2$ these spaces are often denoted by M_ℓ whereas for $d = 3$ the symbol N_ℓ is used. Here $\ell = (l_1, \ldots, l_n)$ is the tuple whose entries are the lengths of the edges of the polygon. These lengths are assumed fixed and their choice determines the topology of the spaces M_ℓ and N_ℓ.

The investigation of the spaces M_ℓ and N_ℓ commenced as part of the study of the topology of configuration spaces of linkages, initiated by W. Thurston and his collaborators ([39]). This investigation was continued in the work of J.-Cl. Hausmann in [20] and of M. Kapovich and J. Millson in [27]. Results on the topology of equilateral polygon spaces were obtained by Y. Kamiyama and his collaborators (see e.g. [25],[26]).

In [40], K. Walker studied the planar polygon spaces M_ℓ, stating a description of the homology groups and a conjecture on the structure of the cohomology rings. A rigorous computation of the homology groups of M_ℓ was given by M. Farber and D. Schuetz in [15]. Walker's conjectural statement was established under an additional assumption by M. Farber, J.-Cl. Hausmann and D. Schuetz ([12]). This assumption was removed in later work of D. Schuetz ([36]), establishing the full statement of the conjecture.

The homology groups of the spaces N_ℓ of spatial polygons were computed by A. A. Klyachko in [30]. In [22], A. Knutson and J.-Cl. Hausmann used the theory of toric varieties to determine the cohomology ring of N_ℓ. An analogue of Walker's conjecture for the spaces N_ℓ was proved in [12].

In this thesis, polygon spaces are studied from a different point of view: we consider polygons up to translations. We denote these *free* polygon spaces by $E_d(\ell)$. Our study of these spaces is motivated by the fact that while the direct analogues of M_ℓ and N_ℓ in dimensions $d > 3$ are singular, the spaces $E_d(\ell)$ are generically manifolds in all dimensions (see Proposition 1.5.3 below).

Chapter 1 is devoted to the discussion of the basic properties of the spaces $E_d(\ell)$. Analogously to the cases of the spaces M_ℓ and N_ℓ, there is a close relationship between the topology of $E_d(\ell)$ and combinatorial properties of the length vector ℓ. In Section 1.3, we recall the combinatorial classification of length vectors in the language of short, long and median sets.

In order to be able to test general results, it is useful to have at one's disposal some cases where the space $E_d(\ell)$ can be determined explicitly. Results of this type are given in Section 1.6 (see also Section 1.2). In Section 1.6, we also introduce and study the robot arm distance map. In our setting, this map is a Morse-Bott function with rather special properties (Lemma 1.6.1).

The main tool behind our results on the homology and the cohomology of the spaces $E_d(\ell)$ is a certain Morse-Bott analogue of the classical Morse lacunary principle. This analogue, which may be of independent interest, is established in Chapter 2; it is applied in Chapter 3 to study the homology groups of the spaces $E_d(\ell)$. The main results are as follows. The $\mathbf{Z}_2$-Betti numbers of the spaces are computed explicitly (Theorem 3.1.1). It is shown that the integral homology groups are torsion-free if d is even; in the case where d is odd, an explicit combinatorial criterion for the existence of torsion elements is found (Theorems 3.1.2, 3.1.3 and 3.1.5). These results complement the previous work of M. Farber and D. Schuetz on the planar case ([15]). Sections 3.2 and 3.3 contain applications and the discussion of several special cases.

In Section 3.4, we pursue the probabilistic approach to the topology of the spaces $E_d(\ell)$: the length vector ℓ is viewed as a random variable and topological invariants of the space $E_d(\ell)$ as random functions. This study is motivated by the idea that in applications the numbers l_j may not be known precisely, but rather different edge lengths arise with different probability. The main results of Section 3.4, Theorems 3.4.1 and 3.4.2, describe the asymptotic behaviour of the homotopy groups and of the Betti numbers of the spaces $E_d(\ell)$ as the number n of edges becomes large. These results are inspired by the work of M. Farber and T. Kappeler in [14] and of M. Farber in [8].

Chapter 4 is concerned with the study of polygonal linkages in the case where the lengths of all the segments except one are fixed and the length of the remaining segment varies in a prescribed interval. Configuration spaces of this type describe mechanisms with a prismatic joint. The main result of Chapter 4 is the computation of the homology groups of these spaces in the planar case. In Section 4.3, we discuss an application motivated by the so-called Topological Hypothesis from the theory of phase transitions.

While the results of Chapter 3 describe the dependence of the homology groups of the spaces $E_d(\ell)$ on the combinatorial properties of the length vector ℓ, the purpose of Chapter 5 is the investigation of the inverse problem. Namely, we study the question whether knowledge of the topology of the space $E_d(\ell)$ is sufficient to determine the length vector ℓ. One of the main results of this thesis, Theorem 5.1.1, states that the graded isomorphism type of the $\mathbf{Z}_2$-cohomology ring $H^*(E_d(\ell); \mathbf{Z}_2)$ determines ℓ up to a combinatorially defined notion of equivalence. This theorem establishes an analogue of Walker's conjecture in all dimensions $d > 2$ and complements the results of [12]. Its proof uses the techniques introduced in [12], with the Morse-Bott lacunary principle of Chapter 2 as a central additional tool.

In a separate appendix we discuss the relationship of the spaces $E_d(\ell)$ to configuration spaces of polygonal chains that were studied in [20] and in [21]. The material of the appendix is motivated by the results in the recent paper [13] and the questions raised therein.

Chapter 1

Spaces of Polygons

In this chapter we discuss basic properties of the spaces of polygons in Euclidean space. We study smoothness of the spaces and determine them explicitly in several special cases. Much of the material of this chapter is expository and similar to previously known results.

1.1 Spaces of Polygons in Euclidean Space

The objective of this thesis is the study of the topology of spaces of polygons in Euclidean space. Throughout, we will denote by $n \geq 3$ the number of edges of the polygon and by $d \geq 2$ the dimension of the ambient Euclidean space.

A polygon whose edges have length $l_1, \ldots, l_n > 0$ is defined by specifying an n-tuple $P_1, \ldots, P_n$ of points in $\mathbb{R}^d$, so that for each $j = 1, \ldots, n-1$ the vector $P_{j+1} - P_j$ has length l_j and the vector $P_1 - P_n$ has length l_n. We will identify two polygons whenever one is obtained from the other by a translation.

The unit vectors in the directions $P_{j+1} - P_j$, $j = 1, \ldots, n-1$ and $P_1 - P_n$ determine the tuple $(P_1, \ldots, P_n)$ uniquely up to translations. Thus formally, the space of polygons may be defined as follows.

Definition 1.1.1. *Let $\ell = (l_1, \ldots, l_n) \in \mathbb{R}^n_{>0}$ be an n-tuple of positive real numbers.*

Denote $W = (S^{d-1})^n$. We define the space of polygons in $\mathbb{R}^d$ on n edges of length $l_1, \ldots, l_n$, viewed up to translations, as the subset $E_d(\ell) \subset W$ given by

$$E_d(\ell) = \{(u_1, \ldots, u_n) \in W : \sum_{j=1}^{n} l_j u_j = 0\},$$

equipped with the subspace topology.

One also refers to $E_d(\ell)$ as the *free* polygon space (in contrast to configuration spaces defined by viewing polygons up to all orientation-preserving Euclidean isometries).

The homeomorphism type of the space $E_d(\ell)$ depends on the choice of the edge lengths $l_1, \ldots, l_n$. We will refer to the tuple $\ell = (l_1, \ldots, l_n)$ as a *length vector.*

As is made more precise below, by varying the length of one of the edges in some interval, one obtains a cobordism between spaces $E_d(\ell)$ for different length vectors ℓ. Formally, such cobordisms are defined as follows.

Let

$$\ell^- = (l_1^-, \ldots, l_n^-)$$

and

$$\ell^+ = (l_1^+, \ldots, l_n^+)$$

be two length vectors so that

$$l_j^- = l_j^+ \text{ for } j = 1, \ldots, n-1 \text{ and } l_n^- < l_n^+.$$

Denote by $A \subset \mathbb{R}^n$ the interval connecting ℓ^- and ℓ^+:

$$A = \{(l_1, \ldots, l_n) \in \mathbb{R}^n : \; l_j^- \leq l_j \leq l_j^+, j = 1, \ldots, n\}.$$

Definition 1.1.2. *For $A \subset \mathbb{R}^n$ as above, we define the space $E_d(A)$ as the subset $E_d(A) \subset W$ given by*

$$E_d(A) = \bigcup_{\ell \in A} E_d(\ell),$$

equipped with the subspace topology.

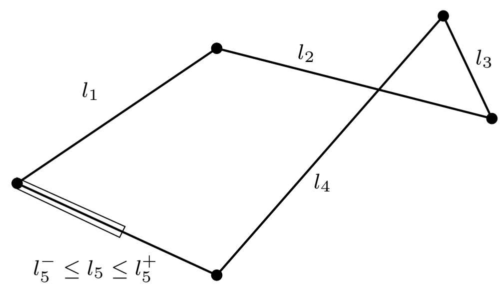

Figure 1.1: A planar polygonal linkage with a prismatic joint.

We can view $E_d(A)$ as the configuration space of a polygonal linkage, where the lengths of all the segments except one are fixed and the remaining segment is *telescopic*: its length varies in a prescribed interval. A segment of this type is sometimes also called a prismatic joint ([33]). We will refer to A as the *metric data* of the telescopic linkage.

1.2 The Case of a small Number of Edges

This section is of mostly expository nature and similar to the material in the first Chapter of [7]. We employ elementary geometric considerations to study the space $E_d(\ell)$ in some simple cases. This approach is useful for illustrating the dependence of the topology of $E_d(\ell)$ on the choice of the length vector ℓ.

Throughout this section, we shall assume that the entries of ℓ are ordered according to their size:

$$l_1 \leq l_2 \leq \cdots \leq l_n. \tag{1.1}$$

We view $W = (S^{d-1})^n$ as the configuration space of a *robot arm* in $\mathbb{R}^d$, a mechanism consisting of n segments connected by revolving joints into a single polygonal chain whose initial point, also referred to as the grip, is fixed at the origin. An element $(u_1, \ldots, u_n) \in W$ corresponds to the configuration where for $j = 1, \ldots, n$, $u_j \in S^{d-1}$ is the direction of the jth segment of the robot arm.

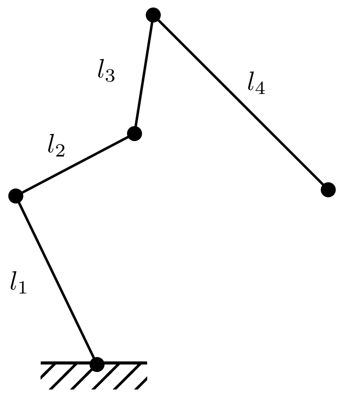

Figure 1.2: A robot arm with four segments.

Denote by

$$\alpha : W \to \mathbb{R}^d$$

the map which associates to every configuration of the robot arm the position of the endpoint. Explicitly, if $l_1, \ldots, l_n$ are the lengths of the individual segments of the robot arm, then the map α is given by

$$\alpha(u_1, \ldots, u_n) = \sum_{j=1}^{n} l_j u_j.$$

The map α is called the *workspace map* and its image the *workspace* of the robot arm. In the case $n = 1$, the workspace is a sphere in $\mathbb{R}^d$. For $n \geq 2$ the workspace is a closed spherical shell $A \subset \mathbb{R}^d$ centred at the origin, with outer radius $r_+ = l_1 + \cdots + l_n$ and inner radius $r_- = max\{l_n - (l_1 + \cdots + l_{n-1}), 0\}$. In particular, in the case of a robot arm with two segments of length $l_1 < l_2$, the workspace is a spherical shell with outer radius $l_1 + l_2$ and inner radius $l_2 - l_1$.

The following lemma summarises some basic properties of the workspace map.

Lemma 1.2.1. *Consider a robot arm with two segments of length $l_1 < l_2$, workspace map $\alpha : W = (S^{d-1})^2 \to \mathbb{R}^d$ and workspace $A = \alpha(W)$.*

1. *For every interior point p of A the preimage $\alpha^{-1}(p)$ is homeomorphic to the sphere S^{d-2}. In particular, in the case $d = 2$ the preimage consists of two points.*

2. *If p is a boundary point of A, then the preimage $\alpha^{-1}(p)$ consists of a single point.*

3. *Let $S \subset \mathbb{R}^d$ be a sphere of radius r centred at the origin. If*

$$l_2 - l_1 < r < l_1 + l_2,$$

then the preimage $\alpha^{-1}(S)$ is homeomorphic to T^1S^{d-1}, the total space of the unit tangent bundle of the sphere S^{d-1}.

For $d = 2$, the space T^1S^{d-1} is a disjoint union of two copies of S^1.

Let us consider the space $E_d(\ell)$ for $n = 3$. We identify elements of $E_d(\ell)$ with configurations of a robot arm with two segments of length l_1 and l_2, so that the endpoint of the arm lies on a circle S of radius l_3 about the origin. Thus in the case $n = 3$ the subset $E_d(\ell) \subset (S^{d-1})^3$ coincides with the preimage $\alpha^{-1}(S)$.

If $l_3 > l_1 + l_2$, then S is disjoint from the workspace of the arm and hence in this case $E_d(\ell) = \emptyset$. If $l_3 < l_1 + l_2$, then using the third part of Lemma 1.2.1, one finds a homeomorphism

$$E_d(\ell) \simeq T^1S^{d-1}.$$

We now study the case $n = 4$, $d = 2$. Here elements of the space $E_2(\ell)$ are quadrangles in the plane, viewed up to translations. Using cyclic notation, we label for $j = 1, 2, 3, 4$ by P_j the vertex of the quadrangle which is adjacent to the edges l_{j-1} and l_j. As two polygons that are obtained from each other by a translation are identified, it may be assumed that one of the vertices P_j lies at some fixed point in $\mathbb{R}^d$. We will assume that the vertex P_2 adjacent to the edges l_1 and l_2 is fixed at the origin.

The geometric locus of the possible positions of the vertex P_4 is the intersection of two annuli $A = A(\rho_1, \rho_2)$ and $A' = A'(\rho'_1, \rho'_2)$ centred at the origin, with inner and outer radii given respectively by

$$\rho_1 = l_3 - l_2, \;\; \rho_2 = l_3 + l_2$$

and by

$$\rho_1' = l_4 - l_1, \ \rho_2' = l_4 + l_1.$$

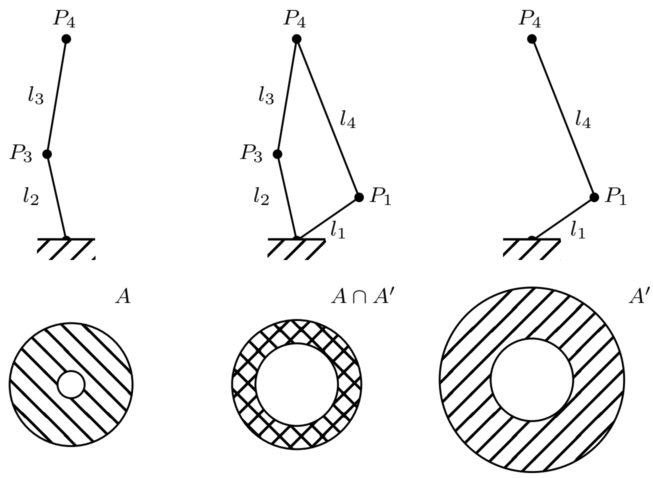

Figure 1.3: The locus of possible positions of the point P_4 as an intersection of workspaces.

If $l_4 > l_1 + l_2 + l_3$, then A and A' are disjoint and hence $E_2(\ell) = \emptyset$.

Next, we assume that $l_4 < l_1 + l_2 + l_3$ and thus the intersection $A \cap A'$ is non-empty. For every interior point P of $A \cap A'$, there are four pairwise distinct quadrangles so that the vertex P_4 lies at P. These four quadrangles are obtained from each other by reflecting the positions of the vertices P_1 and P_3 on the line connecting the origin to P_4. There are two distinct quadrangles so that the vertex P_4 lies at a given boundary point of the intersection $A \cap A'$: in this case either the vertex P_1 or the vertex P_2 lies on the line connecting the origin with P_4.

We see that the space $E_2(\ell)$ is obtained from the disjoint union of four annuli by an identification of boundary components. Since ℓ is ordered,

$$l_4 - l_1 > l_3 - l_2$$

and hence on the inner boundary of $A \cap A'$, the edges l_1 and l_4 are collinear.

If

$$l_2 + l_3 > l_1 + l_4,$$

then on the outer boundary of $A \cap A'$, the edges l_1 and l_4 are collinear as well. Thus in this case the annuli are glued together as indicated by Figure 1.4.

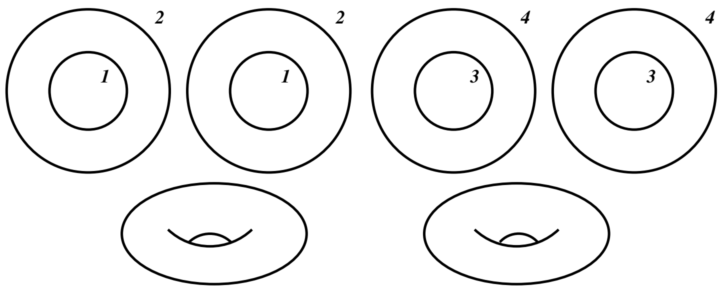

Figure 1.4: The space $E_2(\ell)$ in the case $n = 4$ and $l_2 + l_3 > l_1 + l_4$.

In the case

$$l_2 + l_3 < l_1 + l_4,$$

the edges l_2 and l_3 are collinear on the outer boundary of $A \cap A'$ and the identification is as in Figure 1.5.

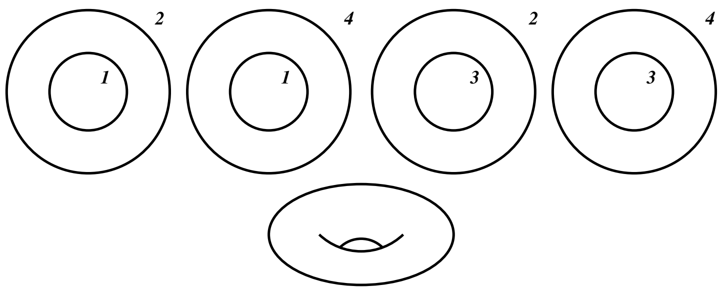

Figure 1.5: The space $E_2(\ell)$ for $n = 4$ and $l_4 - l_1 < l_2 + l_3 < l_1 + l_4$.

We conclude that if $l_2 + l_3 > l_1 + l_4$, then

$$E_2(\ell) \simeq T^2 \cup T^2,$$

whereas for $l_4 - l_1 < l_2 + l_3 < l_1 + l_4$,

$$E_2(\ell) \simeq T^2.$$

The following proposition summarises the above results:

Proposition 1.2.2. *Let ℓ be a length vector with $l_1 \leq l_2 \leq \cdots \leq l_n$.*

1. *Let $n = 3$. If $l_3 > l_1 + l_2$, then*

$$E_d(\ell) = \emptyset,$$

   whereas for $l_3 < l_1 + l_2$, there is a homeomorphism

$$E_d(\ell) \simeq T^1 S^{d-1}.$$

2. *Let $n = 4$ and $d = 2$. If $l_4 > l_1 + l_2 + l_3$, then the space $E_d(\ell)$ is empty.*

   Assume that $l_4 < l_1 + l_2 + l_3$. If $l_2 + l_3 > l_1 + l_4$, then $E_2(\ell)$ is a disjoint union of two tori:

$$E_2(\ell) \simeq T^2 \cup T^2,$$

   whereas for $l_2 + l_3 < l_1 + l_4$, there is a homeomorphism

$$E_2(\ell) \simeq T^2.$$

While similar arguments can be applied to analyze the case $n = 4$, $d > 2$, it will be more convenient to study this case using different methods that are discussed further below. The proof of the following result is given in Section 1.6.

Proposition 1.2.3. *Let $n = 4$ and assume that $l_1 \leq l_2 \leq l_3 \leq l_4$ and $l_4 < l_1 + l_2 + l_3$.*

1. *If $l_2 + l_3 > l_1 + l_4$, then there is a homeomorphism*

$$E_d(\ell) \simeq S^{d-1} \times T^1 S^{d-1}.$$

2. *In the case $l_2 + l_3 < l_1 + l_4$ the space $E_d(\ell)$ is given by*

$$E_d(\ell) \simeq S^{2(d-1)-1} \times S^{d-1}.$$

Since for $d = 2$, the space $T^1 S^{d-1}$ is a disjoint union of two copies of S^1, in this case the claim of Proposition 1.2.3 follows from the second assertion of Proposition 1.2.2.

1.3 Combinatorics of Length Vectors

As illustrated by the discussion in the previous section, there is a close relationship between the topology of the spaces $E_d(\ell)$ and the combinatorial properties of the length vector ℓ. The purpose of this section is to recall the classification of length vectors in the language of short, long and median sets.

A length vector ℓ is called *ordered* if the inequalities

$$l_1 \leq \cdots \leq l_n$$

hold. We call ℓ *generic* if there is no choice of $\epsilon_j = \pm 1$ for $j = 1, \ldots, n$ so that

$$\sum_{j=1}^{n} \epsilon_j l_j = 0.$$

A subset $J \subset \{1, \ldots, n\}$ is called *long* (respectively *short* or *median*) with respect to ℓ if

$$\ell_J = \sum_{j \in J} l_j - \sum_{j \notin J} l_j > 0$$

(respectively. $\ell_J < 0$ or $\ell_J = 0$). A length vector ℓ is generic if and only if no subset $J \subset \{1, \ldots, n\}$ is median with respect to ℓ. The complement

$$\overline{J} = \{1, \ldots, n\} - J$$

of a long set J is a short set.

We write $\mathcal{L}(\ell)$ (respectively $\mathcal{S}(\ell)$, $\mathcal{M}(\ell)$) for the set of all subsets of $\{1, \ldots, n\}$ that are long (respectively short, median) with respect to ℓ. Denote by $\mathcal{L}_n(\ell) \subset \mathcal{L}(\ell)$ (respectively $\mathcal{S}_n(\ell) \subset \mathcal{S}(\ell)$, $\mathcal{M}_n(\ell) \subset \mathcal{M}(\ell)$) those long sets (respectively. short, median sets) which contain the index n.

Definition 1.3.1. *Let ℓ, ℓ' be two length vectors. We say that ℓ and ℓ' lie in the same stratum if the sets $\mathcal{L}(\ell)$ and $\mathcal{L}(\ell')$ coincide. If, in addition, both ℓ and ℓ' are generic, then we say that ℓ and ℓ' lie in the same chamber.*

The geometric meaning of these notions is as follows. For every subset $J \subset \{1, \ldots, n\}$, consider the hyperplane $H_J \subset \mathbb{R}^n$ defined by

$$H_J = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{j \in J} x_j - \sum_{j \notin J} x_j = 0\},$$

also referred to as the *wall* corresponding to J. Since every subset $J \subset \{1, \ldots, n\}$ defines the same wall as its complement, there are 2^{n-1} distinct walls. For $J = \emptyset$ and $J = \{1, \ldots, n\}$, the hyperplane $H_J \subset \mathbb{R}^n$ is disjoint from $\mathbb{R}^n_{>0}$.

A length vector $\ell \in \mathbb{R}^n_{>0}$ is generic if and only if ℓ does not lie on a wall. The walls define a stratification of $\mathbb{R}^n_{>0}$: for every $k \geq 0$ the codimension k strata are the connected components of the subset consisting of those points of $\mathbb{R}^n_{>0}$, which lie on exactly k distinct walls. The codimension zero strata are the chambers. Two generic length vectors lie in the same chamber if and only if they lie in the same connected component of the complement $\mathbb{R}^n_{>0} - \cup_J H_J$ of the union of all the walls.

The following observation is well-known (Lemma 4 in [12], see also Proposition 2.5 in [22]):

Proposition 1.3.2. *Two length vectors ℓ and ℓ' lie in the same stratum if and only if the sets $\mathcal{L}_n(\ell)$ and $\mathcal{S}_n(\ell)$ coincide with $\mathcal{L}_n(\ell')$ and $\mathcal{S}_n(\ell')$ correspondingly.*

If ℓ is generic, then a subset $J \subset \{1, \ldots, n\}$ with $n \in J$ is an element of $\mathcal{S}_n(\ell)$ if and only if $J \notin \mathcal{L}_n(\ell)$. Thus Proposition 1.3.2 implies:

Corollary 1.3.3. *Two generic length vectors ℓ and ℓ' lie in the same chamber if and only if the sets $\mathcal{L}_n(\ell)$ and $\mathcal{L}_n(\ell')$ coincide.*

The following result is useful in understanding how the stratum of an ordered length vector changes when one permutes the entries of the length vector.

Lemma 1.3.4. *Let ℓ, ℓ' be two ordered length vectors.*

1. *If there exists a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ so that $\sigma(n) = n$ and the sets $\sigma(\mathcal{L}_n(\ell))$ and $\mathcal{L}_n(\ell')$ coincide, then the sets $\mathcal{L}_n(\ell)$ and $\mathcal{L}_n(\ell')$ coincide*

as well. Similarly, if $\sigma(n) = n$ and the sets $\sigma(\mathcal{S}_n(\ell))$ and $\mathcal{S}_n(\ell')$ coincide, then so do the sets $\mathcal{S}_n(\ell)$ and $\mathcal{S}_n(\ell')$.

2. *For a permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$, denote by $\sigma(\ell)$ the length vector obtained by interchanging the entries of ℓ by σ. If the length vectors $\sigma(\ell)$ and ℓ' lie in the same stratum, then so do ℓ and ℓ'.*

Proof. The first assertion can be found as Lemma 3 in [12]. The proof given in [12] also shows that if ℓ and ℓ' are ordered and $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ is an arbitrary permutation so that the sets $\sigma(\mathcal{L}(\ell))$ and $\mathcal{L}(\ell')$ coincide, then the sets $\mathcal{L}(\ell)$ and $\mathcal{L}(\ell')$ coincide as well. Since $\sigma(\mathcal{L}(\ell)) = \mathcal{L}(\sigma(\ell))$, this proves the second assertion. □

If we vary an entry of a generic length vector in a sufficiently small interval, we obtain length vectors that lie in the same chamber. To make this statement precise, it is useful to consider for a given length vector ℓ the quantity

$$[\ell] = \min\left(\sum_{j=1}^{n} \epsilon_j l_j\right),$$

where the minimum is taken over all choices of $\epsilon_j \in \{\pm 1\}, j = 1, \dots, n$, so that

$$\sum_{j=1}^{n} \epsilon_j l_j > 0.$$

Proposition 1.3.5. *Let ℓ be a length vector and let $0 < \varepsilon < \min(l_n, [\ell])$. Then*

$$\ell^- = (l_1, \dots, l_{n-1}, l_n - \varepsilon)$$

and

$$\ell^+ = (l_1, \dots, l_{n-1}, l_n + \varepsilon)$$

are both generic. Moreover, if ℓ is generic, then the length vectors ℓ, ℓ^- and ℓ^+ all lie in the same chamber.

Proof. The first assertion readily follows from the definition of the quantity $[\ell]$. If ℓ is generic, then a subset $J \subset \{1, \dots, n\}$ with $n \in J$ is long with respect to ℓ^+ if and only if it is long with respect to ℓ. Thus the sets $\mathcal{L}_n(\ell)$ and $\mathcal{L}_n(\ell^+)$ coincide; analogously so do the sets $\mathcal{L}_n(\ell)$ and $\mathcal{L}_n(\ell^-)$. It follows from Corollary 1.3.3 that in this case ℓ, ℓ^- and ℓ^+ lie in the same chamber. □

One concludes from the definition of the quantity $[\ell]$ that if $0 < \varepsilon < [\ell]$, then every length vector obtained from ℓ by inserting ε as a new entry is generic. The following Lemma, first proved in [23], will be important for the results of Chapter 5:

Lemma 1.3.6 ([23], Lemma 5.1)**.** *Let ℓ, ℓ' be two length vectors and suppose that $0 < \varepsilon < [\ell]$ and $0 < \varepsilon' < [\ell']$. Denote by (ε, ℓ) the length vector obtained from ℓ by inserting ε as the new first entry. If (ε, ℓ) and (ε', ℓ') lie in the same chamber, then ℓ and ℓ' lie in the same stratum.*

Proof. For every subset $J \subset \{1, \ldots, n\}$, denote by $\widetilde{J} \subset \{2, \ldots, n+1\}$ the set $\widetilde{J} = \{j+1 : j \in J\}$. Using the definition of the quantity $[\ell]$, one concludes that the map $J \mapsto \widetilde{J}$ defines a bijection between $\mathcal{L}(\ell)$ and

$$\{J \in \mathcal{L}(\varepsilon, \ell) : 1 \notin J\}.$$

Thus if the two sets $\mathcal{L}(\varepsilon, \ell)$ and $\mathcal{L}(\varepsilon', \ell')$ coincide, then so do the two sets $\mathcal{L}(\ell)$ and $\mathcal{L}(\ell')$. □

1.4 Counting short and median Sets

In this section we define several numerical quantities based on counts of subsets of the index set $\{1, \ldots, n\}$ that are short or median with respect to a given length vector. These numerical quantities will be used in the subsequent study of the Betti numbers of the spaces introduced in Section 1.1.

Definition 1.4.1. *Let ℓ be a length vector and let $m \in \{1, \ldots, n\}$ be the index of any maximal entry of the length vector ℓ. For $k = 0, \ldots, n-1$, we define numbers $a_k(\ell)$, $b_k(\ell)$ and $\alpha_k(\ell)$ as follows:*

1. *We define $a_k(\ell)$ (respectively $b_k(\ell)$) as the number of subsets $J \subset \{1, \ldots, n\}$ so that $m \in J$, $|J| = k+1$ and J is short (respectively median) with respect to ℓ.*

2. *We denote by $\alpha_k(\ell)$ the number of subsets $J \subset \{1, \ldots, n\}$ of cardinality $|J| = k+1$ which contain the index n and are short with respect to ℓ.*

It will be convenient to extend this Definition to the case $k = -1$ by defining $a_{-1}(\ell) = b_{-1}(l) = \alpha_{-1}(\ell) = 0$ for every length vector ℓ.

The difference between the numbers $a_k(\ell)$ and $\alpha_k(\ell)$ is that while the first is the number of short sets which contain the index of a maximal entry of ℓ, the second is the number of short sets containing the index n. If n is the index of a maximal entry of ℓ, then $a_k(\ell)$ and $\alpha_k(\ell)$ coincide for all k.

The proposition below summarises some properties of the numbers $a_k(\ell)$, $b_k(\ell)$ and $\alpha_k(\ell)$ that are readily concluded from their Definition:

Proposition 1.4.2. *1. A length vector ℓ is generic if and only if all the numbers $b_k(\ell)$ vanish.*

2. If the length vector ℓ' is obtained from ℓ by a permutation of the entries, then the numbers $a_k(\ell)$ and $a_k(\ell')$ coincide for all k. If ℓ' is obtained from ℓ by a permutation which fixes the last entry, then $\alpha_k(\ell) = \alpha_k(\ell')$ for all k.

The following numbers play an important role in the study of the homology of spaces of polygons with a telescopic edge.

Definition 1.4.3. *Let $\ell^- = (l_1, \ldots, l_{n-1}, l_n^-)$ and $\ell^+ = (l_1, \ldots, l_{n-1}, l_n^+)$ be two length vectors with*

$$l_1 \leq \cdots \leq l_{n-1}.$$

For $k = 0, \ldots, n-1$, we define $\beta_k(\ell^+, \ell^-)$ to be the number of subsets $J \subset \{1, \ldots, n\}$ which satisfy the following conditions:

1. *$|J| = k + 1$.*
2. *$n \in J$ and $n - 1 \notin J$.*
3. *J is short with respect to ℓ^-.*
4. *The subset $J' = J - \{n\} \cup \{n - 1\}$ is long with respect to ℓ^+.*

It will also be useful to extend this definition to the case $k = -1$ by setting $\beta_{-1}(\ell^+, \ell^-) = 0$ for all ℓ^+, ℓ^-.

The next proposition describes the basic properties of the numbers β_k.

Proposition 1.4.4. *1. For $k = 0, \ldots, n-2$,*

$$\beta_k(\ell^-, \ell^+) = \beta_{n-k-2}(\ell^+, \ell^-).$$

2. If

$$l_n^- + l_n^+ \geq 2l_{n-1},$$

then $\beta_k(\ell^+, \ell^-) = 0$ for all k.

3. Let ℓ be generic and let

$$\ell^- = (l_1, \ldots, l_{n-1}, l_n - \varepsilon)$$

and

$$\ell^+ = (l_1, \ldots, l_{n-1}, l_n + \varepsilon),$$

where $0 < \varepsilon < \min(l_n, [\ell])$. Then

$$\beta_k(\ell^+, \ell^-) = \alpha_k(\ell) - a_k(\ell).$$

Proof. If a subset $J \subset \{1, \ldots, n\}$ satisfies the four conditions of Definition 1.4.3, then the set

$$I = \overline{J} \cup \{n\} - \{n-1\}$$

satisfies conditions 2,3 and 4 of this Definition, with the roles of ℓ^- and ℓ^+ interchanged. Here $\overline{J}$ denotes the complement of J in $\{1, \ldots, n\}$. If $|J| = k+1$, then the cardinality of the set I is

$$|I| = n - k - 1 = (n-k-2) + 1.$$

We conclude that $\beta_k(\ell^-, \ell^+) = \beta_{n-k-2}(\ell^+, \ell^-)$.

To prove the second claim, let $J \subset \{1, \ldots, n\}$ be a subset satisfying the first three conditions of Definition 1.4.3. Consider the set $J' = J - \{n\} \cup \{n-1\}$. The quantity $\ell^+_{J'}$ can be expressed as

$$\ell^+_{J'} = \ell^+_J - 2l^+_n + 2l_{n-1} = \ell^-_J - (l^-_n + l^+_n) + 2l_{n-1}.$$

Thus if $l^-_n + l^+_n \geq 2l_{n-1}$, then the inequality

$$\ell^+_{J'} \leq \ell^-_J \leq 0$$

holds and the fourth condition of Definition 1.4.3 is violated. We see that if $l^-_n + l^+_n \geq 2l_{n-1}$, then no subset $J \subset \{1, \ldots, n\}$ satisfies the four conditions of Definition 1.4.3 simultaneously. Thus in this case $\beta_k(\ell^+, \ell^-) = 0$ for all k.

We now assume that ℓ^- and ℓ^+ are length vectors as in the third claim of the proposition. Since ℓ, ℓ^- and ℓ^+ lie in the same chamber,

$$\alpha_k(\ell^-) = \alpha_k(\ell^+) = \alpha_k(\ell) \text{ and } \beta_k(\ell^-, \ell^+) = \beta_k(\ell, \ell)$$

for all k. It follows from Definition 1.4.1 that if $l_{n-1} \leq l_n$, then the numbers $\alpha_k(\ell)$ and $a_k(\ell)$ coincide for all k. On the other hand, by the second part of the proposition, in this case all the numbers $\beta_k(\ell, \ell)$ vanish. This proves the third claim of the proposition in the case $l_{n-1} \leq l_n$.

Assume now the inequality $l_{n-1} > l_n$. We write the set $\mathcal{S}_n(\ell)$ as a disjoint union

$$\mathcal{S}_n(\ell) = A \cup B \cup C$$

of subsets of the following form:

- A is the family of all subsets $J \subset \{1, \ldots, n\}$ so that $n-1, n \in J$ and J is short with respect to ℓ.
- B is the family of all subsets $J \subset \{1, \ldots, n\}$ so that $n \in J$, $n-1 \notin J$ and the sets J and $J - \{n\} \cup \{n-1\}$ are both short with respect to ℓ.
- C is the family of all subsets $J \subset \{1, \ldots, n\}$ so that $n-1 \notin J$, J is short with respect to ℓ and the set $J - \{n\} \cup \{n-1\}$ is long with respect to ℓ.

Since $l_{n-1} > l_n$, for every subset $J \subset \{1, \ldots, n\}$ which is short with respect to ℓ and satisfies $n-1 \in J$ and $n \notin J$, the set $J - \{n-1\} \cup \{n\}$ is again short with respect to ℓ. Thus the set B is in bijection with the set of all $J \subset \{1, \ldots, n\}$ so that $n-1 \in J$, $n \notin J$ and J is short with respect to ℓ. It follows that the number of elements $J \in A \cup B$ with $|J| = k+1$ is $a_k(\ell)$. On the other hand, one concludes from Definitions 1.4.1 and 1.4.3 that the number of elements $J \in S_n(\ell)$ (respectively $J \in C$) with $|J| = k+1$ is $\alpha_k(\ell)$ (respectively $\beta_k(\ell, \ell)$). It follows that $\alpha_k(\ell) = a_k(\ell) + \beta_k(\ell, \ell)$. □

1.5 Basic Properties of the Spaces

In this section we relate basic properties of the spaces $E_d(\ell)$ and $E_d(A)$ such as non-emptiness and smoothness to combinatorial properties of length vectors.

Proposition 1.5.1. *1. Let ℓ be a length vector and let $m \in \{1, \ldots, n\}$ be the index of any maximal entry of ℓ. The space $E_d(\ell)$ is non-empty if and only if the one-element set $\{m\}$ is short or median with respect to ℓ.*

2. Consider a polygonal telescopic linkage with metric data A given by length vectors $\ell^- = (l_1, \ldots, l_{n-1}, l_n^-)$ and $\ell^+ = (l_1, \ldots, l_{n-1}, l_n^+)$. Let $k \in \{1, \ldots, n-1\}$ be the index of any maximal entry of $(l_1, \ldots, l_{n-1})$. The space $E_d(A)$ is non-empty if and only if the one-element set $\{n\}$ is short or median with respect to ℓ^- and the one-element set $\{k\}$ is short or median with respect to ℓ^+.

Proof. It follows from the definition of the space $E_d(\ell)$ and the triangle inequality that a necessary condition for $E_d(\ell)$ to be non-empty is that the one-element set $\{m\}$ be short or median with respect to ℓ. By the first part of Proposition 1.2.2, this condition is also sufficient if $n = 3$. Let us show that the condition is sufficient for every $n \geq 3$.

Assume that the set $\{m\}$ is short or median with respect to ℓ. Let $J \subset \{1, \ldots, n\}$ be a maximal subset satisfying the conditions

$$m \notin J \text{ and } J \text{ is short with respect to } \ell.$$

Then $J \neq \emptyset$ since otherwise every one-element set $\{k\} \subset \{1, \dots, n\}$ would be long or median with respect to ℓ, contradicting the assumption $n \geq 3$. Moreover, $\overline{J} \neq \{m\}$ since the complement of a short subset is long. It follows that the set $I = \overline{J} - \{m\}$ is non-empty.

We note that I is either short or median with respect to ℓ. Indeed, if I were long, then using the fact that m is the index of a maximal entry of ℓ, one would obtain that for every $i \in I$ the set $J \cup \{i\}$ is short with respect to ℓ, contradicting the choice of J. We have found a partition

$$\{1, \dots, n\} - \{m\} = I \cup J$$

so that $I, J \neq \emptyset$ and the sets I, J are both short or median with respect to ℓ.

The length vector L with entries

$$L_1 = \sum_{j \in J} l_j,\ L_2 = \sum_{j \in I} l_j \text{ and } L_3 = l_m$$

satisfies the condition of the first part of the proposition with $n = 3$, as every one-element subset $\{j\} \subset \{1, 2, 3\}$ is either short or median with respect to L. We conclude that $E_d(L) \neq \emptyset$. On the other hand, every element of the space $E_d(L)$ may be viewed as a polygon in $E_d(\ell)$ so that all the edges with indices in I are collinear and so are all the edges with indices in J. Thus $E_d(\ell) \neq \emptyset$.

We now demonstrate the second assertion of the proposition. If $E_d(A) \neq \emptyset$, then there exists a length vector $\ell = (l_1, \dots, l_n) \in A$ with $E_d(\ell) \neq \emptyset$. Using the first part of the proposition, it follows that each of the sets $\{k\}$ and $\{n\}$ is either short or median with respect to ℓ. Since $l_n^- \leq l_n \leq l_n^+$, one concludes that the set $\{n\}$ is short or median with respect to ℓ^- and the set $\{k\}$ is short or median with respect to ℓ^+.

We now show that if $E_d(A) = \emptyset$, then one of the two sets $\{k\}$, $\{n\}$ must be long with respect to both ℓ^- and ℓ^+. This will complete the proof.

If the space $E_d(A)$ is empty, then so is the space $E_d(\ell^-)$. Using the first part of the proposition, it follows that in this case one of the sets $\{k\}, \{n\}$ is long with respect to ℓ^-. If the set $\{n\}$ is long with respect to ℓ^-, then it is also long with respect to ℓ^+. If $\{k\}$ were long with respect to ℓ^- but short or median with respect to ℓ^+, there would exist $\ell \in A$ so that $\{k\}$ is median with respect to ℓ. But then by the first part of the proposition $E_d(\ell) \neq \emptyset$ contradicting the assumption that the space $E_d(A)$ is empty. $\square$

It follows from Definition 1.1.1 that the subset $E_d(\ell) \subset W$ is invariant under the diagonal action of the orthogonal groups $SO(d)$ and $O(d)$ on $W = (S^{d-1})^n$.

Proposition 1.5.2. *1. The $SO(2)$-action on the space $E_2(\ell)$ is free. If M_ℓ denotes the quotient space*

$$M_\ell = E_2(\ell)/SO(2),$$

then there is a homeomorphism

$$E_2(\ell) \simeq S^1 \times M_\ell.$$

2. The action of $SO(3)$ on $E_3(\ell)$ is free if and only if the length vector ℓ is generic.

3. If $d > 3$ and $E_d(\ell) \neq \emptyset$, then the $SO(d)$-action on $E_d(\ell)$ is not free.

Proof. For $d = 2$, the group $SO(2)$ acts freely on $W = (S^1)^n$ and hence also on $E_2(\ell) \subset W$. The space M_ℓ can be identified with the subset of $E_2(\ell)$ consisting of those tuples $u = (u_1, \ldots, u_n) \in E_2(\ell) \subset W$, where $u_1 = e$ is a given fixed element of S^1. A homeomorphism $\phi : S^1 \times M_\ell \to E_2(\ell)$ is defined by $\phi(\theta, u) = (\theta e, \theta u_2, \ldots, \theta u_n)$ for $\theta \in S^1$ and $u = (e, u_2, \ldots, u_n) \in M_\ell$.

For $d = 3$, the isotropy subgroup of an element $(u_1, \ldots, u_n) \in (S^{d-1})^n = W$ is non-trivial if and only if $u_j = \pm u_k$ for all $j, k \in \{1, \ldots, n\}$. The subset $E_d(\ell) \subset W$ contains elements $(u_1, \ldots, u_n)$ of this form if and only if the length vector ℓ is non-generic.

If $d > 3$, then every planar configuration in $E_d(\ell)$ (that is a configuration where the edges of the polygon lie in a two-dimensional subspace $E \subset \mathbb{R}^d$) has a non-trivial isotropy subgroup, since such a configuration is fixed by every element of $SO(d)$ which fixes E. It follows from the proof of Proposition 1.5.1 that if the space $E_d(\ell)$ is non-empty, then it contains a planar configuration. $\square$

Proposition 1.5.3. *1. Let ℓ and ℓ' be two length vectors, so that ℓ' can be obtained from ℓ by a permutation of the entries. Then the spaces $E_d(\ell)$ and $E_d(\ell')$ may be naturally identified.*

2. If the length vector ℓ is generic, then the space $E_d(\ell)$ is a closed oriented manifold of dimension

$$\dim\ E_d(\ell) = (d-1)(n-1) - 1.$$

3. If two generic length vectors ℓ and ℓ' lie in the same chamber, then the spaces $E_d(\ell)$ and $E_d(\ell')$ are $O(d)$-equivariantly diffeomorphic.

To prove Proposition 1.5.3, consider $F : (\mathbb{R}^d)^{n-1} \to \mathbb{R}^n$,

$$F(v_1, \ldots, v_{n-1}) = (|v_1|, |v_2 - v_1|, \ldots, |v_{n-1} - v_{n-2}|, |v_{n-1}|).$$

The map F is continuous and its restriction to the open subset $X \subset (\mathbb{R}^d)^{n-1}$ given by the inequalities

$$v_1 \neq 0,\ v_{n-1} \neq 0 \text{ and } v_{j+1} \neq v_j \text{ for } j = 1, \ldots, n-2$$

is smooth. Moreover, $F(X) = \mathbb{R}^n_{>0}$.

Lemma 1.5.4. *Every generic length vector is a regular value of $F : X \to \mathbb{R}^n_{>0}$.*

Proof. We will show that the critical points of $F|_X$ are exactly those tuples

$$v = (v_1, \ldots, v_{n-1}) \in X,$$

so that all the vectors $v_j \in \mathbb{R}^d$, $j = 1, \ldots, n-1$ are collinear. Since for a length vector $\ell \in \mathbb{R}^n_{>0}$ the preimage $F^{-1}(\ell)$ contains points v of this form if and only if ℓ is non-generic, this will imply the claim.

Let $V = (V_1, \ldots, V_{n-1}) \in T_v(\mathbb{R}^d)^{n-1}$. Denote

$$u_1 = \frac{1}{l_1} v_1, \ u_n = -\frac{1}{l_n} v_{n-1}$$

$$u_j = \frac{1}{l_j}(v_j - v_{j-1}), \ 2 \leq j \leq n-1$$

and write $\langle \cdot, \cdot \rangle$ for the standard scalar product on $\mathbb{R}^d$. The derivative $D_V F$ of F in direction V is the vector in $\mathbb{R}^n$ with entries

$$(D_V F)_1 = \langle V_1, u_1 \rangle,$$

$$(D_V F)_j = \langle V_j - V_{j-1}, u_j \rangle, \ j = 2, \ldots, n-1$$

and

$$(D_V F)_n = -\langle V_{n-1}, u_n \rangle.$$

Let v be a critical point of F. Then there exists

$$\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{R}^n, \ \epsilon \neq 0,$$

so that the image of the derivative of F at v is contained in the hyperplane orthogonal to ϵ. Thus in this case

$$\sum_{j=1}^{n-1} \langle V_j, \epsilon_j u_j - \epsilon_{j+1} u_{j+1} \rangle = 0$$

for all $V_1, \ldots, V_{n-1}$. This implies that $u_1, \ldots, u_n$ satisfy

$$\epsilon_j u_j = \epsilon_{j+1} u_{j+1}$$

for $j = 1, \ldots, n-1$. Thus the vectors $u_1, \ldots, u_n$, and hence also $v_1, \ldots, v_{n-1}$ are collinear. □

Proof of Proposition 1.5.3. The first assertion follows from the fact that a permutation σ of the indices $1, \ldots, n$ defines a diffeomorphism $W \to W$ which maps $E_d(\ell)$ onto $E_d(\sigma(\ell))$.

Identifying

$$E_d(\ell) \simeq F^{-1}(\ell),$$

the second claim follows from Lemma 1.5.4.

To prove the third assertion, let ℓ and ℓ' be two generic length vectors which lie in the same chamber. Denote by $[\ell, \ell'] \subset \mathbb{R}^n$ the closed interval connecting ℓ and ℓ'. Since every length vector $\ell'' \in [\ell, \ell']$ is generic, $F^{-1}([\ell, \ell'])$ is a smooth cobordism between $E_d(\ell)$ and $E_d(\ell')$. As

$$F : F^{-1}([\ell, \ell']) \to [\ell, \ell']$$

is a smooth function without critical points, this cobordism is trivial and therefore there is a diffeomorphism $E_d(\ell) \simeq E_d(\ell')$.

Since F is invariant under the diagonal action of the orthogonal group $O(d)$ on $(\mathbb{R}^d)^{n-1}$, so is the projection $\pi : F^{-1}([\ell, \ell']) \to I$. Consider the flow of the gradient of π with respect to the metric on $F^{-1}([\ell, \ell']) \subset (\mathbb{R}^d)^{n-1}$ induced by the Euclidean metric. The flow defines a diffeomorphism $E_d(\ell) \to E_d(\ell')$ and since both π and the metric are $O(d)$-invariant, this is an equivariant diffeomorphism. □

We now state the analogue of Proposition 1.5.3 for polygons with a telescopic edge.

We shall call the metric data A of the telescopic linkage generic if both length vectors ℓ^- and ℓ^+ are generic.

Proposition 1.5.5. *Suppose that A is generic.*

1. *The space $E_d(A)$ is a compact oriented manifold of dimension*

$$dim\ E_d(A) = (d-1)(n-1).$$

   The boundary of $E_d(A)$ is given by the disjoint union of the spaces $E_d(\ell^-)$ and $E_d(\ell^+)$.

2. *Let $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation which fixes the index n. Let $\sigma(A)$ be the metric data of the polygonal telescopic linkage described by the length vectors $\sigma(\ell^-)$ and $\sigma(\ell^+)$. Then the spaces $E_d(A)$ and $E_d(\sigma(A))$ are $O(d)$-equivariantly diffeomorphic.*

Proposition 1.5.6. *Let $\ell = (l_1, \ldots, l_n)$ be generic, let $0 < \varepsilon < \min(l_n, [\ell])$ and let*

$$\ell^- = (l_1, \ldots, l_{n-1}, l_n - \varepsilon),$$

$$\ell^+ = (l_1, \ldots, l_{n-1}, l_n + \varepsilon)$$

be length vectors as in Proposition 1.3.5. Let A be the metric data of the polygonal telescopic linkage corresponding to ℓ^- and ℓ^+. Then there is an $O(d)$-equivariant diffeomorophism

$$E_d(A) \simeq E_d(\ell) \times [-\varepsilon, \varepsilon].$$

The proofs of Propositions 1.5.5 and 1.5.6 are given in the next section.

1.6 The Robot Arm Distance Map

In this section we introduce the robot arm distance map and study its properties. We determine the space $E_d(\ell)$ explicitly in some important special cases.

The proofs of many subsequent results concerning the topology of the spaces $E_d(\ell)$ rely on the special properties of a certain function on $W = (S^{d-1})^n$ that we now introduce.

Lemma 1.6.1. *Let ℓ be a length vector. Consider the function*

$$f_\ell : W \to \mathbb{R},\ f_\ell(u_1, \ldots, u_n) = -\Big|\sum_{j=1}^{n} l_j u_j\Big|^2.$$

1. *The critical points of f_ℓ are given by the union of $f_\ell^{-1}(0) = E_d(\ell)$ and all the submanifolds*

   $$P_J = \{(u_1, \ldots, u_n) :\ u_j = u_k = -u_m \text{ for } j, k \in J, m \notin J\},$$

   so that the subset $J \subset \{1, \ldots, n\}$ is long with respect to ℓ.

2. *For every subset $J \subset \{1, \ldots, n\}$ which is long with respect to ℓ, the critical submanifold P_J is nondegenerate in the sense of Bott. Its Morse-Bott index is given by*

   $$ind_{f_\ell}(P_J) = (d-1)(n - |J|).$$

Remark 1.6.2. The function f_ℓ is called the *robot arm distance map.* Recall from Section 1.2 that $W = (S^{d-1})^n$ may be viewed as the configuration space of a robot arm in Euclidean space of dimension d. If the segments of the robot arm have length $l_1, \dots, l_n$, then the function f_ℓ measures the distance between the grip and the endpoint of the arm. The space $E_d(\ell)$ is the subset of W consisting of those configurations of the robot arm, so that the endpoint of the arm lies at the grip.

The first part of Lemma 1.6.1 states that the critical points of f_ℓ consist of the zero level set $f_\ell^{-1}(0) = E_d(\ell)$ as well as of all those configurations of the robot arm, so that all the segments are collinear.

The second part of Lemma 1.6.1 implies that the restriction of f_ℓ to the complement $W - E_d(\ell)$ is a Morse-Bott function with the property that every critical submanifold is diffeomorphic to the sphere S^{d-1} and has a Morse-Bott index which is a multiple of $d-1$. In the next chapter, we develop an analogue of the classical Morse lacunary principle that applies to Morse-Bott functions of this type.

Proof. For $u \in S^{d-1}$, identify

$$T_u S^{d-1} \simeq \{v \in \mathbb{R}^d : \langle u, v\rangle = 0\}.$$

Let $(v_1, \dots, v_n) \in T_{(u_1,\dots,u_n)}W$. For $j = 1, \dots, n$, the derivative of f_ℓ in direction v_j is

$$D_{v_j} f_\ell = -2\langle l_j v_j, \sum_{i=1,\dots,n} l_i u_i\rangle.$$

At a critical point $(u_1, \dots, u_n)$ of f_ℓ, the equation $D_{v_j} f_\ell = 0$ holds for $j = 1, \dots, n$. It follows that if $f_\ell(u_1, \dots, u_n) \neq 0$, then for each $j = 1, \dots, n$ the vector u_j is collinear to the sum $\sum_{i=1,\dots,n-1} l_i u_i$. Thus the critical points of f_ℓ are those tuples $(u_1, \dots, u_n)$, where either $f_\ell(u_1, \dots, u_n) = 0$ or $u_j = \pm u_k$ for all j, k. This proves the first assertion.

To establish the second assertion, fix an element $p \in S^{d-1}$ and consider for every subset $J \subset \{1, \dots, n\}$ which is long with respect to ℓ the point $p_J \in P_J$ given by

$$u_j = \begin{cases} p & \text{if } j \in J, \\ -p & \text{if } j \notin J. \end{cases}$$

We want to compute explicitly the Hessian of f_ℓ at the critical point p_J. By symmetry, it suffices to consider the case $p = e_1 = (1, 0, \dots, 0)$. Parametrize S^{d-1} in a neighbourhood of e_1 by the map

$$\mathbb{R}^{d-1} \to S^{d-1} \subset \mathbb{R}^d,$$

$$(r_2, \dots, r_d) \mapsto \frac{1}{(1 + \sum_{2 \le j \le d} r_j^2)^{1/2}} \Big(e_1 + \sum_{2 \le j \le d} r_j e_j \Big)$$

and in a neighbourhood of $-e_1$ by

$$\mathbb{R}^{d-1} \to S^{d-1} \subset \mathbb{R}^d,$$

$$(r_2, \dots, r_d) \mapsto -\frac{1}{(1 + \sum_{2 \le j \le d} r_j^2)^{1/2}} \Big(e_1 + \sum_{2 \le j \le d} r_j e_j \Big).$$

We can express the second derivatives of f_ℓ at the point p_J as follows:

$$\frac{\partial^2}{\partial r_{k_1}^{(j_1)} \partial r_{k_2}^{(j_2)}} f_\ell = \begin{cases} -2l_j^2 + 2\epsilon_J(j) l_j \ell_J & \text{if } (k_1, j_1) = (k_2, j_2) = (k, j), \\ -2 l_{j_1} l_{j_2} & \text{if } k_1 = k_2, j_1 \neq j_2, \\ 0 & \text{if } k_1 \neq k_2. \end{cases}$$

Here $k_1, k_2 \in \{2, \dots, d\}$, $j_1, j_2 \in \{1, \dots, n\}$ and

$$\epsilon_J(j) = \begin{cases} +1 & \text{if } j \in J, \\ -1 & \text{if } j \notin J \end{cases}.$$

We also recall the notation

$$\ell_J = \sum_{j \in J} l_j - \sum_{j \notin J} l_j.$$

It follows that the Hessian of f_ℓ at p_J is congruent to a $(d-1)n \times (d-1)n$-matrix A of the following form. Let $D \in \mathbb{R}^{n \times n}$ be the diagonal matrix whose jth diagonal entry for $j = 1, \dots, n$ is

$$(D)_{jj} = \frac{\epsilon_J(j)}{l_j} \ell_J$$

and $E \in \mathbb{R}^{n\times n}$ be the matrix with all entries equal to 1. Then A is obtained from the difference $D - E$ by interchanging every entry λ with the matrix λI_{d-1} where $I_{d-1} \in \mathbb{R}^{(d-1)\times(d-1)}$ is the identity matrix. Thus the Hessian of f_ℓ has the same eigenvalues as the matrix $D-E$ and the multiplicity of every eigenvalue is multiplied by $(d-1)$. Using the computation in [7], Lemma 1.4, the index of the Hessian is $(n-|J|)(d-1)$ and the multiplicity of the zero eigenvalue is $d-1$. Since $d-1$ is also the dimension of the critical submanifold P_J, it follows that it is nondegenerate in the sense of Bott. □

We can now give the proofs of Propositions 1.5.5 and 1.5.6.

Proof of Propositions 1.5.5 and 1.5.6. Let A be the metric data of a telescopic linkage defined by $\ell^- = (l_1, \ldots, l_{n-1}, l_n^-)$ and $\ell^+ = (l_1, \ldots, l_{n-1}, l_n^+)$. Denoting by ℓ' the length vector $\ell' = (l_1, \ldots, l_{n-1})$, the space $E_d(A)$ may be identified with the preimage $f_{\ell'}^{-1}([a,b])$, where

$$a = -(l_n^+)^2 \text{ and } b = -(l_n^-)^2.$$

Moreover, $f_{\ell'}^{-1}(a) = E_d(\ell^+)$ and $f_{\ell'}^{-1}(b) = E_d(\ell^-)$.

By Proposition 1.6.1, the critical values of $f_{\ell'}$ consist of 0 and all the numbers of the form $-|\sum_{j=1}^{n-1} \epsilon_j l_j|^2$, where $\epsilon_j = \pm 1$. Thus a and b are regular values of $f_{\ell'}$ if and only if the length vectors ℓ^- and ℓ^+ are both generic. In this case $E_d(A) = f_{\ell'}^{-1}([a,b])$ is a manifold whose boundary is given by the disjoint union of $f_{\ell'}^{-1}(a) = E_d(\ell^+)$ and $f_{\ell'}^{-1}(b) = E_d(\ell^-)$. This proves the first claim of Proposition 1.5.5. The proof of the second claim is analogous to that of the first part of Proposition 1.5.3.

If ℓ is generic and $\ell^- = (l_1, \ldots, l_{n-1}, l_n - \varepsilon)$, $\ell^+ = (l_1, \ldots, l_{n-1}, l_n + \varepsilon)$, $0 < \varepsilon < \min(l_n, [\ell])$ are length vectors as in Proposition 1.3.5, then every element of A is a generic length vector. Thus in this case the function $f_{\ell'}$ has no critical values in the interval $[a,b]$ and hence there are diffeomorphisms

$$E_d(A) \simeq f_{\ell'}^{-1}([a,b]) \simeq f_{\ell'}^{-1}(-l_n^2) \times [a,b] \simeq E_d(\ell) \times [a,b].$$

Since the function $f_{\ell'}$ is $O(d)$-invariant, the same argument as used in the proof of the first part of Proposition 1.5.3 shows that the diffeomorphism $E_d(A) \simeq E_d(\ell) \times [a, b]$ may be chosen to be $O(d)$-equivariant. □

Let ℓ be a generic length vector. We say that $J \subset \{1, \ldots, n\}$ is a *maximal short subset* if J is short with respect to ℓ and if for every proper inclusion $J \subsetneq K \subset \{1, \ldots, n\}$ the subset K is long with respect to ℓ. We say that there is a *massive edge* if there exists a maximal short one-element subset $\{m\} \subset \{1, \ldots, n\}$. Using the numerical quantities $a_k(\ell)$ introduced in Section 1.4, this condition can equivalently be expressed by the two equations

$$a_0(\ell) = 1,\ a_1(\ell) = 0.$$

Proposition 1.6.3 (The case of a massive edge)**.** *Suppose that a generic length vector ℓ admits a maximal short one-element subset $\{m\} \subset \{1, \ldots, n\}$. If $n = 3$, then there is a diffeomorphism*

$$E_d(\ell) \simeq T^1 S^{d-1}.$$

If $n > 3$, then the space $E_d(\ell)$ is diffeomorphic to a product of two spheres:

$$E_d(\ell) \simeq S^{(d-1)(n-2)-1} \times S^{d-1}.$$

Remark 1.6.4. The assumption of Proposition 1.6.3 determines the chamber of the length vector ℓ uniquely up to permutation of the entries of ℓ. Indeed, in this case a subset $J \subset \{1, \ldots, n\}$ is short with respect to ℓ if either $J = \{m\}$ or $m \notin J$ and $|J| < n - 1$, and long otherwise.

Proof of Proposition 1.6.3. Using the first part of Proposition 1.5.3, we can assume without loss of generality that $m = n$ is the index of the last entry of ℓ. Denote by $\ell' = (l_1, \ldots, l_{n-1})$ the length vector obtained from ℓ by erasing the last entry and consider the function $f_{\ell'} : (S^{d-1})^{n-1} \to \mathbb{R}$. We identify

$$E_d(\ell) = f_{\ell'}^{-1}(-l_n^2).$$

The minimal value if $f_{\ell'}$ is $-L^2$, where $L = l_1 + \cdots + l_{n-1}$. The preimage $f_{\ell'}^{-1}(-L^2)$ is the diagonal $\Delta \subset (S^{d-1})^{n-1}$. By Lemma 1.6.1, $f_{\ell'}^{-1}(-L^2)$ is a critical submanifold which is nondegenerate in the sense of Bott. Moreover, critical values of $f_{\ell'}$ which lie in the interval $[-L^2, -l_n^2]$ correspond to subsets of $\{1, \ldots, n\}$ which are long with respect to ℓ and do not containing the index n. Since under the assumptions of the proposition the only subset of this form is $\{1, \ldots, n-1\}$, we find that $-L^2$ is the only critical value contained in this interval. It follows that $E_d(\ell)$ is diffeomorphic to the space of unit vectors in the normal bundle ν to the diagonal $\Delta \subset (S^{d-1})^{n-1}$. To identify this latter space, we note that the bundle ν is isomorphic to the Whitney sum of $n-2$ copies of the tangent bundle of the sphere S^{d-1}:

$$\nu \simeq \bigoplus_{n-2} TS^{d-1}.$$

It follows that if $n = 3$, then there is a diffeomorphism

$$E_d(\ell) \simeq T^1 S^{d-1}.$$

Assume now that $n > 3$. We claim that in this case the vector bundle ν is trivial. To prove this, the criterion of Theorem 1.5 in Chapter 9 of [24] will be used. Let τ denote the tangent bundle of S^{d-1} and θ^k the trivial bundle of rank k over S^{d-1}. Since the Whitney sum $\tau \oplus \theta^1$ is trivial, so is

$$\nu \oplus \theta^{n-2} = \oplus_{n-2}(\tau \oplus \theta).$$

Since for $n > 3$ the rank $(d-1)(n-2)$ of the bundle ν is greater than the dimension $d-1$ of its base, the aforementioned theorem in [24] shows that triviality of $\nu \oplus \theta^{n-2}$ implies the triviality of ν. It follows that in the case $n > 3$ there is a diffeomorphism

$$E_d(\ell) \simeq S^{(d-1)(n-2)-1} \times S^{d-1}.$$

This completes the proof. $\square$

Let us now study how the space $E_d(\ell)$ changes when a small edge is inserted into the polygon.

Proposition 1.6.5 (Inserting a small edge)**.** *Let ℓ be a generic length vector and let $0 < \varepsilon < [\ell]$ (see Section 1.3 for the definition of the quantity $[\ell]$). Denote by (ε, ℓ) the length vector obtained from ℓ by inserting ε as the first entry. Then there is a diffeomorphism*

$$E_d(\varepsilon, \ell) \simeq S^{d-1} \times E_d(\ell).$$

Proof. Consider the map

$$\widehat{F} : (\mathbb{R}^d)^n \to \mathbb{R}^d \times \mathbb{R}^n,$$

$$(v_1, \ldots, v_n) \mapsto (v_1, |v_2 - v_1|, \ldots, |v_n - v_{n-1}|, |v_n|).$$

We note that $\widehat{F}$ is smooth on the open subset $Y \subset (\mathbb{R}^d)^n$ defined by the inequalities

$$v_n \neq 0 \text{ and } v_{j+1} \neq v_j \text{ for } j = 1, \ldots, n-1.$$

It follows from the proof of Lemma 1.5.4 that every critical point of $\widehat{F}|_Y$ is a tuple $(v_1, \ldots, v_n)$ so that all the vectors v_j, $j = 1, \ldots, n$ are collinear. Thus every vector of the form $(v_1, \ell) \in \mathbb{R}^d \times \mathbb{R}^n$, $0 < |v_1| < [\ell]$, is a regular value of $\widehat{F}|_Y$. Moreover, since ℓ is generic, the vector $(0, \ell)$ is a regular value of $\widehat{F}|_Y$.

Denote by $B_\varepsilon(0) \subset \mathbb{R}^d$ the closed ball of radius ε centred at the origin. We identify

$$E_d(\varepsilon, \ell) = \widehat{F}^{-1}(\partial B_\varepsilon(0) \times \{\ell\}).$$

Since the map $\widehat{F}$ has no critical values in $B_\varepsilon(0) \times \{\ell\}$, there are diffeomorphisms

$$E_d(\varepsilon, \ell) \simeq \widehat{F}^{-1}(\partial B_\varepsilon(0) \times \{\ell\}) \simeq S^{d-1} \times \widehat{F}^{-1}(0, \ell) \simeq S^{d-1} \times E_d(\ell).$$

□

Let ℓ be a length vector. We say that there is a *massive triangle* if there are pairwise distinct indices $i, j, k \in \{1, \ldots, n\}$ so that the two-element sets $\{i, j\}$, $\{i, k\}$ and $\{j, k\}$ are long with respect to ℓ. In this case one of the indices i, j, k must be the index of a maximal entry of ℓ. Moreover, a subset $J \subset \{1, \ldots, n\}$ is long with respect to ℓ if J contains at least two of the indices i, j, k, and short otherwise. Thus existence of a massive triangle implies that the length vector ℓ is generic and determines its chamber uniquely up to permutation of the entries of ℓ.

Existence of indices i, j, k as above is equivalent to the number $a_{n-3}(\ell)$ being nonzero. Indeed, it follows from Definition 1.4.1 that $a_{n-3}(\ell)$ is the number of long two-element subsets of $\{1, \ldots, n\}$ not containing the index of a maximal entry. Thus $a_{n-3}(\ell) \neq 0$ if there is a massive triangle. Conversely, if $a_{n-3}(\ell) \neq 0$, then there exists a long subset $J = \{j_1, j_2\} \subset \{1, \dots, n\}$ not containing the index m of the maximal entry, but then the three indices j_1, j_2, m satisfy the above condition.

Proposition 1.6.6 (The case of a massive triangle)**.** *If $a_{n-3}(\ell) \neq 0$, then ℓ is generic and there is a diffeomorphism*

$$E_d(\ell) \simeq (S^{d-1})^{n-3} \times T^1 S^{d-1}.$$

Remark 1.6.7. The number $a_{n-3}(\ell)$ is either zero or one. Indeed, suppose that there existed two distinct long two-element sets $J = \{j_1, j_2\}$ and $K = \{k_1, k_2\}$ not containing the index m of a maximal entry of ℓ. Since the complement of a long subset is short, the intersection of J and K must be non-empty and we can assume that $j_1 = k_1 = j$. Since $J \neq K$, the indices j_2 and k_2 must be distinct. As $l_n \geq l_j$, we find that the two disjoint sets $\{j, j_2\}$ and $\{k_2, m\}$ are both long. This is a contradiction.

Proof of Proposition 1.6.6. Using the first part of Proposition 1.5.3, we can assume without loss of generality that the length vector ℓ is ordered. If $n = 3$, then the diffeomorophism $E_d(\ell) \simeq T^1 S^{d-1}$ follows from Proposition 1.6.3. Assume now that $n > 3$.

Since ℓ is generic, by slightly changing one of the entries of ℓ if necessary, we may assume that the length vector $\ell' = (l_2, \dots, l_n)$ obtained by erasing the first entry of ℓ is generic. Let $0 < \varepsilon < [\ell']$. We claim that the length vectors ℓ and (ε, ℓ') lie in the same chamber. Indeed, a subset $J \subset \{1, \dots, n\}$ is long with respect to ℓ if and only if J contains at least two of the three indices $n-2, n-1, n$ and this last condition is equivalent to J being long with respect to (ε, ℓ). Using Propositions 1.5.3 and 1.6.5, it follows that

$$E_d(\ell) \simeq S^{d-1} \times E_d(\ell').$$

We note that the set $\{n-3, n-2\}$ is long with respect to ℓ'. Thus, arguing inductively, one finds $E_d(\ell) \simeq (S^{d-1})^{n-3} \times T^1 S^{d-1}$. This completes the proof. $\square$

We can now conclude Proposition 1.2.3 as a corollary of Propositions 1.6.3 and 1.6.6. Indeed, in the case of the first part of Proposition 1.2.3 we have $a_1(\ell) \neq 0$ and the homeomorphism $E_d(\ell) \simeq S^{d-1} \times T^1 S^{d-1}$ follows from Proposition 1.6.6. In the situation of the second part of Proposition 1.2.3, the assumptions of Proposition 1.6.3 are satisfied and thus $E_d(\ell) \simeq S^{2(d-1)-1} \times S^{d-1}$.

We finish this chapter by using the robot arm distance map to study connectedness of the spaces $E_d(\ell)$.

Proposition 1.6.8. *Assume that for some $k \geq 1$ the length vector ℓ does not admit long subsets $J \subset \{1, \dots, n\}$ with $|J| = k$. Then the homomorphism*

$$\pi_p(E_d(\ell)) \to \pi_p((S^{d-1})^n)$$

induced by inclusion is an isomorphism for $0 \leq p < (d-1)k - 1$ and an epimorphism for $p = (d-1)k - 1$.

Proof. Using Proposition 1.5.3, we can assume without loss of generality that the length vector ℓ is ordered. Consider the robot arm distance map $f_\ell : (S^{d-1})^n \to \mathbb{R}$ defined in Proposition 1.6.1. Recall that the space $E_d(\ell)$ is the zero level set $f_\ell^{-1}(0)$.

Denote $t_0 = 0$ and let $t_1 < t_0$ be a regular value of f_ℓ so that the interval (t_1, t_0) contains no critical values. Moreover, let $t_1 > t_2 > \cdots > t_s$ be regular values of f_ℓ so that for each $i > 0$ the interval (t_{i+1}, t_i) contains a single critical value of f_ℓ and so that the image of f_ℓ is contained in the interval $[t_0, t_s)$. Consider the preimage $W_i = f_\ell^{-1}[t_i, t_0]$ for $i > 0$. Since $t_0 = 0$ is the maximum of f_ℓ and since every t_i with $i > 0$ is a regular value of f_ℓ, it follows that each subset $W_i \subset W$ is a manifold with boundary. The gradient flow of f_ℓ defines a deformation retraction of W_1 onto $E_d(\ell) = f_\ell^{-1}(0)$. The choice of t_s implies that $W_s = (S^{d-1})^n$.

By Lemma 1.6.1, the critical points of f_ℓ contained in $W_{i+1} - W_i$ lie on the submanifolds $P_J \subset (S^{d-1})^n$, where $J \subset \{1, \dots, n\}$ is long with respect to ℓ and $f_\ell(P_J) \in (t_{i+1}, t_i)$. Every critical submanifold P_J is nondegenerate in the sense of Bott, homeomorphic to the sphere S^{d-1} and has the Morse-Bott index

$$\mathrm{ind}_{f_\ell}(P_J) = (d-1)(n - |J|).$$

Denote by E_J^- the unstable bundle of P_J with respect to a gradient flow of f_ℓ and let $D_J^- \subset E_J^-$ be the corresponding closed disk bundle. Thus the fibre of D_J^- is a disk of dimension $(d-1)(|J|-1)$.

Applying Theorem 2.43 from [34], one concludes that the inclusion into W_{i+1} of the space

$$W_i \bigcup_{f_\ell(P_J)\in(t_{i+1},t_i)} (\cup_{\partial D_J^-} D_J^-) \tag{1.2}$$

obtained by attaching to W_i the unstable disk bundles of all the critical submanifolds P_J with $f_\ell(P_J) \in (t_{i+1}, t_i)$ is a homotopy equivalence. Using the cell decomposition of $P_J \simeq S^{d-1}$ into one zero-cell and one cell of dimension $(d-1)$, it follows that W_{i+1} is homotopy equivalent to a space obtained from W_i by attaching for every subset $J \subset \{1,\ldots,n-1\}$ with $f_\ell(P_J) \in (t_{i+1}, t_i)$ a cell of dimension $(d-1)(|J|-1)$ and then attaching to the resulting space a cell of dimension $(d-1)|J|$.

The above arguments show that if every subset $J \subset \{1,\ldots,n\}$ which is long with respect to ℓ has cardinality $|J| > k$, then each manifold W_{i+1} is homotopy equivalent to a space obtained from W_i by attaching cells of dimension at least $(d-1)k$. Thus in this case the homomorphism $\pi_p(W_i) \longrightarrow \pi_p(W_{i+1})$ induced by inclusion is an isomorphism for $p < (d-1)k - 1$ and an epimorphism for $p = (d-1)k - 1$ (see e.g. Corollary 4.12 in [19]). Arguing inductively, one concludes that the inclusion homomorphism $\pi_p(W_1) \longrightarrow \pi_p(W_s)$ is an isomorphism for $p < (d-1)k - 1$ and an epimorphism for $p = (d-1)k-1$. Since $W_s = (S^{d-1})^n$ and since $E_d(\ell) \subset W_1$ is a deformation retract, this completes the proof. □

It follows from Proposition 1.5.1 that if the space $E_d(\ell)$ is non-empty, then every subset $J \subset \{1,\ldots,n\}$ which is long with respect to ℓ has at least two elements. Together with Proposition 1.6.8, we conclude:

Corollary 1.6.9. *If $d > 2$, then the space $E_d(\ell)$ is $(d-3)$-connected. In particular, in this case the space $E_d(\ell)$ is connected.*

The second part of Proposition 1.2.2 shows that the space $E_d(\ell)$ is not $(d-2)$-connected in general. However, one can show:

Proposition 1.6.10. *Let $d \geq 2$. If $a_{n-3}(\ell) = 0$ (compare with Proposition 1.6.6), then the space $E_d(\ell)$ is $(d-2)$-connected. In particular, in this case the space $E_2(\ell)$ is connected.*

Proof. For every length vector ℓ, there is a fibration $E_d(\ell) \to S^{d-1}$ whose fibre $C_d(\ell)$ is the configuration space of polygonal chains with segment lengths $l_1, \ldots, l_n$ (see Appendix A). It was shown in Proposition 2.7 of [21] that if $a_{n-3}(\ell) = 0$, then the space $C_d(\ell)$ is $(d-2)$-connected. The claim of the Proposition follows using the homotopy exact sequence of the fibration $E_d(\ell) \to S^{d-1}$. $\square$

Chapter 2

Homology and Morse-Bott Theory

The main result of this chapter is a certain generalisation to the Morse-Bott case of the classical Morse lacunary principle. This result will be used in subsequent chapters to study the homology and the cohomology of the spaces $E_d(\ell)$.

2.1 A brief Exposition of the Morse Complex

The purpose of this section is to give a brief account of the construction of the Morse complex. We refer to [34] and to [37] for detailed expositions.

Let M be a closed manifold. For a Morse function f on M, define $C_*(f; \mathbf{Z})$ as the free abelian group generated by the critical points of f, graded by their Morse index.

Let $\mathrm{Crit}(f)$ denote the set of critical points of f. Recall that a vector field X is *gradient-like* with respect to f if

$$X(f) > 0 \text{ on } M - \mathrm{Crit}(f)$$

and if, moreover, for every critical point p of f there exist local coordinates $(x_1, \ldots, x_n)$ near p, so that $x_j(p) = 0$ for $j = 1, \ldots, n$ and

$$f(x_1, \ldots, x_n) = f(p) - \sum_{j=1}^{k} x_j^2 + \sum_{j=k+1}^{n} x_j^2$$

as well as

$$X(x_1,\dots,x_n) = -2\sum_{j=1}^{k} x_j \frac{\partial}{\partial x_j} + 2\sum_{j=k+1}^{n} x_j \frac{\partial}{\partial x_j}.$$

Here n denotes the dimension of M and k the Morse index of the critical point p.

For $p, q \in \mathrm{Crit}(f)$, denote by $U_p(f)$ (respectively by $S_q(f)$) the unstable manifold at p (respectively the stable manifold at q) of the flow of $-X$. We say that a vector field X on M is a Morse-Smale vector field adapted to f if X is gradient-like with respect to f and if, moreover, all the intersections

$$U_p(f) \cap S_q(f),\ \ p, q \in \mathrm{Crit}(f)$$

are transverse. It is well-known that such a vector field exists for every Morse function (see e.g. [34], Theorem 2.27).

For each pair p, q of critical points of f, elements of the space

$$\widetilde{\mathcal{M}}(p,q) = U_p(f) \cap S_q(f)$$

may be viewed as trajectories $\gamma : \mathbb{R} \to M$ of the flow of $-X$ satisfying

$$\lim_{t\to-\infty} \gamma(t) = p \text{ and } \lim_{t\to\infty} \gamma(t) = q.$$

For every element $\gamma \in \widetilde{\mathcal{M}}(p,q)$ and each $s \in \mathbb{R}$ the map

$$\gamma_s : \mathbb{R} \to M,\ \gamma_s(t) = \gamma(s+t)$$

is also an element of $\widetilde{\mathcal{M}}(p,q)$. Thus $\widetilde{\mathcal{M}}(p,q)$ is equipped with a free $\mathbb{R}$-action and one defines the space $\mathcal{M}(p,q)$ of unparametrised flow lines originating in p and ending in q as the quotient

$$\mathcal{M}(p,q) = \widetilde{\mathcal{M}}(p,q)/\mathbb{R}.$$

If X is a Morse-Smale vector field adapted to f, then the space $\mathcal{M}(p,q)$ is empty if $|p| - |q| - 1 < 0$ and a manifold of dimension

$$\dim \mathcal{M}(p,q) = |p| - |q| - 1$$

if $|p| - |q| - 1 \geq 0$.

Choosing orientations of the unstable manifolds of all the critical points defines orientations of all the manifolds $\mathcal{M}(p,q)$. If X is a Morse-Smale vector field adapted to f, then for every pair $p, q \in \mathrm{Crit}(f)$ with $|p| - |q| = 1$, the space $\mathcal{M}(p,q)$ is compact. One defines

$$\partial : C_*(f;\mathbf{Z}) \to C_{*-1}(f;\mathbf{Z}) \tag{2.1}$$

by

$$\partial p = \sum_{|p|-|q|=1} |\mathcal{M}(p,q)| q.$$

Here $|\mathcal{M}(p,q)|$ denotes the number of elements of $\mathcal{M}(p,q)$, counted with orientation. To demonstrate that $\partial^2 = 0$ and thus ∂ defines the structure of a chain complex on $C_*(f;\mathbf{Z})$, one studies a natural compactification of the components $\mathcal{M}(p,q)$ with $|p| - |q| = 2$.

If $H_*(f;\mathbf{Z})$ denotes the homology of the complex $C_*(f;\mathbf{Z})$, then there is an isomorphism

$$H_*(f;\mathbf{Z}) \simeq H_*(M;\mathbf{Z}). \tag{2.2}$$

We mention some known extensions of the construction of the Morse complex $C_*(f;\mathbf{Z})$. If one uses the flow of X rather than the flow of $-X$, then, repeating the above steps in the construction of the complex $(C_*(f), \delta)$, one obtains a cochain complex $(C^*(f), d)$ whose cohomology is isomorphic to the cohomology $H^*(M;\mathbf{Z})$ of M. If M is compact but $\partial M \neq \emptyset$, then the complex $C_*(f;\mathbf{Z})$ is well-defined if $df \neq 0$ on the boundary of M and if ∂M coincides with the set of points of M where f attains its maximum. Moreover, in this case the isomorphism (2.2) continues to hold.

One finds in the literature several approaches which generalise the construction of the complex $C_*(f;\mathbf{Z})$ to the case of Morse-Bott functions (see [2], [3], as well as Appendix A in [17]).

2.2 A lacunary Principle for Morse-Bott Functions

In this section we recall the classical Morse lacunary principle and study a generalisation to the case of Morse-Bott functions.

Let f be a Morse function on a compact manifold M. If M has non-empty boundary, assume that $df \neq 0$ on ∂M and that ∂M coincides with the set of points of M, where f attains its maximum.

We say that f is a *lacunary* Morse function if the indices of all the critical points of f are multiples of some integer $k \geq 2$. A Morse function f is called *perfect* if there are isomorphisms

$$C_*(f; \mathbf{Z}) \simeq H_*(M; \mathbf{Z})$$

of abelian groups.

The Morse lacunary principle states that every lacunary Morse function is perfect. This claim is readily confirmed using the picture of the Morse complex given in the previous section: in the case where f is lacunary, the differential (2.1) vanishes identically for dimensional reasons and perfectness of f follows from (2.2).

We now propose a generalisation of the lacunary principle to the Morse-Bott case. This generalisation is motivated by the special properties of the robot arm distance map that were demonstrated in Lemma 1.6.1.

We recall that a smooth function $f : M \to \mathbb{R}$ is called Morse-Bott if the set $\mathrm{Crit}(f)$ of critical points of f is a disjoint union of submanifolds of M and the restriction of the Hessian of f to each connected component C of $\mathrm{Crit}(f)$ is nondegenerate in normal directions. For every connected component $C \subset \mathrm{Crit}(f)$, denote by $\mathrm{ind}_f(C)$ the Morse-Bott index of C. A Morse-Bott function f is called perfect if there are isomorphisms

$$H_*(M;\mathbf{Z}) \simeq \bigoplus_{C\subset \mathrm{Crit}(f)} H_{*-\mathrm{ind}_f(C)}(M;\mathbf{Z}) \tag{2.3}$$

of abelian groups. The direct sum on the right-hand side of (2.3) is over the connected components C of the set of critical points of f.

Definition 2.2.1. *Let f be a Morse-Bott function. We say that f is lacunary if there exists an integer $k \geq 2$ so that the following conditions are satisfied:*

1. *The Morse-Bott index $ind_f(C)$ of every connected component $C \subset Crit(f)$ is a multiple of k.*

2. *The homology groups $H_*(C;\mathbf{Z})$ of every component are free abelian and the non-trivial groups are concentrated in dimensions which are multiples of k.*

Theorem 2.2.2 (A lacunary principle for Morse-Bott functions)**.** *Let f be a smooth function on a compact manifold M. If $\partial M \neq \emptyset$, then assume that $df \neq 0$ on ∂M and that the boundary ∂M coincides with the set of points of M where f attains its maximum.*

If f is lacunary in the sense of Definition 2.2.1, then f is perfect. Thus in this case there are isomorphisms of abelian groups

$$H_*(M;\mathbf{Z}) \simeq \bigoplus_C H_{*-ind_f(C)}(C;\mathbf{Z}),$$

where the direct sum is over the connected components C of the set of critical points of f.

Proof. Denote for $a \in \mathbb{R}$ by M^a the preimage $M^a = f^{-1}(-\infty, a)$. We want to show that for every $a \in \mathbb{R}$, there is an isomorphism

$$H_*(M^a;\mathbf{Z}) \simeq \bigoplus_{f(C)<a} H_{*-\mathrm{ind}_f(C)}(C;\mathbf{Z}).$$

If the interval $(-\infty, a)$ contains exactly one critical value of f, then existence of this isomorphism follows from the fact that the union of all the critical submanifolds C, for which $f(C) < a$, is a deformation retract of M^a. Assume now that the desired isomorphism has been established for some $a \in \mathbb{R}$ and that for some $b > a$ there is exactly one critical value of f in the interval (a, b).

The unstable bundle of every critical submanifold C is orientable since the cohomology group $H^1(C; \mathbf{Z}_2)$ vanishes due to the second condition of Definition 2.2.1. Using excision and the Thom isomorphism theorem, it follows that

$$H_*(M^b, M^a; \mathbf{Z}) \simeq \bigoplus_{f(C)\in(a,b)} H_{*-\mathrm{ind}_f(C)}(C; \mathbf{Z}). \tag{2.4}$$

In particular, the non-trivial homology groups $H_*(M^b, M^a; \mathbf{Z})$ are concentrated in dimensions which are multiples of k.

Arguing inductively, we can assume that the non-trivial groups $H_*(M^a; R)$ lie in dimensions that are multiples of k. Then the homological exact sequence of the pair (M^b, M^a) reduces to short exact sequences

$$0 \mapsto H_*(M^a; \mathbf{Z}) \to H_*(M^b; R) \to H_*(M^b, M^a; \mathbf{Z}) \to 0. \tag{2.5}$$

Using (2.4) and the second assumption of Definition 2.2.1, the relative groups $H_*(M^b, M^a; \mathbf{Z})$ are free abelian. Thus the short exact sequence (2.5) splits, yielding an isomorphism

$$\begin{aligned} H_*(M^b; R) &\simeq H_*(M^a; \mathbf{Z}) \oplus H_*(M^b, M^a; \mathbf{Z}) \\ &\simeq H_*(M^a; \mathbf{Z}) \oplus \bigoplus_{f(C)\in(a,b)} H_{*-\mathrm{ind}_f(C)}(C; \mathbf{Z}). \end{aligned}$$

This completes the proof. $\square$

2.3 Identifying a Homology Basis

In this section we consider the case of a lacunary Morse-Bott function (see Definition 2.2.1) with the additional property that every component $C \subset \mathrm{Crit}(f)$ is homeomorphic to the sphere S^k. One concludes from Theorem 2.2.2 that under this additional assumption each critical submanifold C contributes two generators to the homology of M, in dimensions $\mathrm{ind}_f(C)$ and $\mathrm{ind}_f(C)+k$. The following criterion can be used to identify these generators.

Theorem 2.3.1. *Let M be a compact manifold and $f : M \to \mathbb{R}$ a Morse-Bott function. If $\partial M \neq \emptyset$, then assume that $df \neq 0$ on ∂M and that ∂M coincides with the set of points where f attains its maximum.*

Assume that f is lacunary (see Definition 2.2.1) and that every component $C \subset Crit(f)$ is homeomorphic to the k-dimensional sphere. Moreover, suppose that for each component C there are closed submanifolds

$$V_C, W_C \subset M$$

with the following properties:

1. *dim $W_C = ind_f(C) + k$, $C \subset W_C$ and $f(p) > f(q)$ for $p \in C$, $q \in W_C - C$.*
2. *$C \subset W_C$ is a nondegenerate critical submanifold of the restriction of f to W_C.*
3. *dim $V_C = ind_f(C)$ and $V_C \subset W_C$. Furthermore C and V_C intersect transversally as submanifolds of W_C in one point.*

Then the collection of the homology classes $\{[V_C], [W_C]\}$, where C is a connected component of Crit(f), forms a basis of $H_*(M; \mathbf{Z}_2)$.

Proof. We use notation from the proof of Theorem 2.2.2. We want to show that for every $a \in \mathbb{R}$, there is a basis of $H_*(M^a; \mathbf{Z}_2)$ consisting of the classes $[V_C], [W_C]$ for $f(C) < a$.

We assume first that the interval $(-\infty, a)$ contains a single critical value of f. Then there is a deformation retraction of M^a onto the union of those components C of the set of critical points of f, where f attains its minimum. For every such component C,

$$W_C = C \text{ and } V_C = \{pt\} \in C.$$

Since C is homeomorphic to the sphere, the homology classes of W_C and V_C form a basis of $H_*(C; \mathbf{Z}_2)$. It follows that the classes $[V_C], [W_C]$ with $f(C) < a$ form a homology basis of $H_*(M^a; \mathbf{Z}_2)$.

Suppose now that the statement has been proved for some $a \in \mathbb{R}$ and that the interval (a, b) contains a single critical value of f. We want to show that there is a basis of $H_*(M^b; \mathbf{Z}_2)$ consisting of the classes $[V_C], [W_C]$ for $f(C) < b$. By the proof of Theorem 2.2.2, there is an isomorphism

$$H_*(M^b; \mathbf{Z}_2) \simeq H_*(M^a; \mathbf{Z}_2) \oplus H_*(M^b, M^a; \mathbf{Z}_2). \tag{2.6}$$

Thus it suffices to show that the relative homology classes $[W_C, W_C \cap M^a]$ and $[V_C, V_C \cap M^a]$, where $f(C) \in (a, b)$, form a basis of $H_*(M^b, M^a; \mathbf{Z}_2)$. Using excision, the deformation retraction given by the negative gradient flow of f and the Thom isomorphism,

$$H_*(M^b, M^a; \mathbf{Z}_2) \simeq \bigoplus_{f(C) \in (a,b)} H_*(W_C, W_C \cap M^a; \mathbf{Z}_2) \tag{2.7}$$

$$\simeq \bigoplus_{f(C) \in (a,b)} H_{*-\mathrm{ind}_f(C)}(C; \mathbf{Z}_2). \tag{2.8}$$

The images under the composition of the two isomorphisms (2.7) and (2.8) of the classes $[W_C, W_C \cap M^a]$ and $[V_C, V_C \cap M^a]$ are respectively $[W_C \cap C] = [C]$ and $[V_C \cap C] = [pt]$. Since every critical submanifold C is homeomorphic to the sphere, these images form a basis of $H_*(C; \mathbf{Z}_2)$. Thus the classes $[W_C, W_C \cap M^a]$ and $[V_C, V_C \cap M^a]$, where $f(C) \in (a, b)$, form a basis of $H_*(M^b, M^a; \mathbf{Z}_2)$. This completes the proof. □

If in the situation of Theorem 2.3.1 the submanifolds $V_C, W_C \subset M$ are all oriented, then the claim of the proposition continues to hold when homology with $\mathbf{Z}_2$-coefficients is replaced by integral homology:

Corollary 2.3.2. *Suppose that in addition to the assumptions of Proposition 2.3.1, for every component $C \subset Crit(f)$ the submanifolds $V_C, W_C \subset M$ are oriented. Then the collection of the homology classes $\{[V_C], [W_C]\}$, where $C \subset Crit(f)$ is a connected component, forms a free basis of $H_*(M; \mathbf{Z})$.*

Proof. It suffices to show that under these additional assumptions the claim of the inductive step of the proof of Theorem 2.3.1 also holds in the case of integral homology.

Since the unstable bundle of each critical submanifold is orientable, we may replace in (2.7) and (2.8) the coefficient group $\mathbf{Z}_2$ by $\mathbf{Z}$. We fix an orientation of the normal bundle to C in W_C. Together with the given orientations of the manifolds W_C and V_C, this defines orientations of the intersections $W_C \cap C = C$ and $V_C \cap C = \{pt\}$, so that the images under the composition of (2.7) and (2.8) of the classes $[W_C, W_C \cap M^a] \in H_{\mathrm{ind}_f(C)+k}(M^b, M^a; \mathbf{Z})$ and $[V_C, V_C \cap M^a] \in H_{\mathrm{ind}_f(C)}(M^b, M^a; \mathbf{Z})$ generate $H_k(C; \mathbf{Z})$ respectively $H_0(C; \mathbf{Z})$. This completes the proof. $\square$

Chapter 3

Homology of Spaces of Polygons

In this chapter we study the homology groups of the spaces $E_d(\ell)$. We explicitly compute the $\mathbf{Z}_2$-Betti numbers, show that the integral homology groups are torsion-free when d is even and find a combinatorial criterion for the existence of torsion elements in the integral homology groups in the case where d is odd. These results cover both generic and non-generic length vectors. We also compute the asymptotic behaviour of the homotopy groups and of the average Betti numbers of the spaces $E_d(\ell)$ as the number n of edges becomes large.

3.1 The Homology Groups

By the first part of Proposition 1.5.2, there is a homeomorphism $E_2(\ell) \simeq S^1 \times M_\ell$, where $M_\ell = E_2(\ell)/SO(2)$ is the space of planar polygons, viewed up to translations as well as rotations. The integral homology groups of the spaces M_ℓ were computed in [15]. In this section we present results concerning the homology of the spaces $E_d(\ell)$ in the case $d > 2$.

In Section 1.4, we defined the combinatorial quantities $a_k(\ell)$ and $b_k(\ell)$ as the numbers of subsets $J \subset \{1, \ldots, n\}$ of cardinality $|J| = k + 1$ which contain the index of a maximal entry of the length vector ℓ and are short respectively median with respect to ℓ.

Theorem 3.1.1. *Let $d > 2$. The non-vanishing $\mathbf{Z}_2$-Betti numbers of the spaces $E_d(\ell)$ are*

$$dim_{\mathbf{Z}_2} H_{(d-1)k}(E_d(\ell); \mathbf{Z}_2) = a_k(\ell) + b_k(\ell) + a_{k-1}(\ell) + b_{k-1}(\ell)$$

for $k = 0, \ldots, n-2$ as well as

$$dim_{\mathbf{Z}_2} H_{(d-1)k-1}(E_d(\ell); \mathbf{Z}_2) = a_{n-1-k}(\ell) + a_{n-2-k}(\ell)$$

for $k = 1, \ldots, n-1$.

The proof, given in Section 3.5, relies on the Morse-Bott lacunary principle established in Chapter 2.

It follows from the computation in [15] of the integral homology groups of the spaces M_ℓ, the homeomorphism $E_2(\ell) \simeq S^1 \times M_\ell$ and the Künneth theorem that the integral homology groups of the spaces $E_2(\ell)$ are free abelian. Concerning the integral homology of the spaces $E_d(\ell)$ for $d > 2$, we will show:

Theorem 3.1.2. *Let $d > 2$.*

1. *The non-vanishing homology groups $H_p(E_d(\ell); \mathbf{Z})$ are concentrated in dimensions*

   $$p = (d-1)k, \ 0 \leq k \leq n-2$$

   and

   $$p = (d-1)k - 1, \ 1 \leq k \leq n-1.$$

2. *The groups $H_{(d-1)k}(E_d(\ell); \mathbf{Z})$, $0 \leq k \leq n-2$ are free abelian.*

Next, we study the question of the existence of torsion elements in the groups $H_{(d-1)k-1}(E_d(\ell); \mathbf{Z})$, $1 \leq k \leq n-1$.

Theorem 3.1.3. *If d is even, then the groups*

$$H_{(d-1)k-1}(E_d(\ell); \mathbf{Z}), \ 1 \leq k \leq n-1$$

are free abelian.

Together with Theorem 3.1.2, it follows that all the integral homology groups of the space $E_d(\ell)$ are torsion-free if d is even. Using the universal coefficient theorem, we obtain:

Corollary 3.1.4. *If $d > 2$ is even, then the Betti numbers*

$$b_p(E_d(\ell)) = rk\ H_p(E_d(\ell); \mathbf{Z})$$

coincide with the $\mathbf{Z}_2$-Betti numbers given in Theorem 3.1.1.

The results of Chapter 1 show that if d is odd, then the homology groups of the form $H_{(d-1)k-1}(E_d(\ell); \mathbf{Z})$ in general contain torsion elements. For example, for $d = 3$ and $\ell = (1, 1, 1)$ one concludes from the first part of Proposition 1.2.2 that

$$H_1(E_3(\ell); \mathbf{Z}) \simeq H_1(T^1S^2; \mathbf{Z}) \simeq H_1(SO(3); \mathbf{Z}) \simeq \mathbf{Z}_2.$$

We will show that in the case of odd d, the existence of torsion elements can be detected by an explicit combinatorial criterion:

Theorem 3.1.5. *Assume that d is odd. Let $m \in \{1, \dots, n\}$ be the index of any maximal entry of the length vector ℓ.*

The following conditions are equivalent:

1. *There are torsion elements in the homology group $H_{(d-1)k-1}(E_d(\ell); \mathbf{Z})$, where $k \in \{1, \dots, n-2\}$;*

2. *There exists a subset $J \subset \{1, \dots, n\}$ with the following properties:*

   (a) *$m \notin J$, $|J| = k + 1$ and J is long respect to ℓ.*

   (b) *There exist indices $i, j \in J$, $i \neq j$, so that the set $I = J - \{i, j\} \cup \{m\}$ is either short or median with respect to ℓ.*

In many cases Theorem 3.1.5 can be used to rule out the existence of torsion elements in the integral homology groups of the space $E_d(\ell)$. For example, suppose that $l_m \geq l_i + l_j$ for all of indices $i, j \neq m$, in other words that there is an edge which is no shorter than any two other edges combined.

Using the formula

$$\ell_I = \ell_J + 2(l_m - l_i - l_j),$$

we see that for every long subset $J \subset \{1, \ldots, n\}$ with $m \notin J$ and every pair $i, j \in J$ the set $I = J - \{i, j\} \cup \{m\}$ is again long with respect to ℓ. Thus in this case there are no subsets $J \subset \{1, \ldots, n\}$ that satisfy the conditions of the second part of Theorem 3.1.5. One concludes:

Corollary 3.1.6. *Suppose that there is an edge which is no shorter than any two other edges combined: there exists an index $m \in \{1, \ldots, n\}$ so that*

$$l_m \geq l_i + l_j \text{ for all } i, j \neq m.$$

Then for every $d \geq 2$ the integral homology groups of the space $E_d(\ell)$ are free abelian. Thus if $d > 2$, then the Betti numbers $b_p(E_d(\ell)) = rk\ H_p(E_d(\ell); \mathbf{Z})$ coincide with the $\mathbf{Z}_2$-Betti numbers computed in Theorem 3.1.1.

The proofs of Theorems 3.1.2, 3.1.3 and 3.1.5 are given in Section 3.6.

3.2 Examples and Applications

We now discuss some applications of the results given in the preceding section.

By Theorem 3.1.1, the space $E_d(\ell)$ is connected if $d > 2$ and the homology groups $H_p(E_d(\ell); \mathbf{Z}_2)$ with $1 \leq p < d - 2$ vanish. This is consistent with Corollary 1.6.9 which states that for $d > 2$ the space $E_d(\ell)$ is $(d - 3)$-connected.

The first possible non-vanishing $\mathbf{Z}_2$-Betti number of positive dimension is

$$\dim_{\mathbf{Z}_2} H_{d-2}(E_d(\ell); \mathbf{Z}_2) = a_{n-3}(\ell) + a_{n-2}(\ell).$$

By Definition 1.4.1, $a_{n-2}(\ell)$ is the number of subsets $J \subset \{1, \ldots, n\}$ which contain the index m of a maximal entry of ℓ, are short with respect to ℓ and have cardinality $|J| = n-1$. Since $n \geq 3$, the number $a_{n-2}(\ell)$ vanishes. As was remarked in 1.6.7, the number $a_{n-3}(\ell)$ can only assume the values 0 and 1 and the condition $a_{n-3}(\ell) = 1$ implies that ℓ is generic and determines the chamber of ℓ uniquely up to permutation of the entries. Using Proposition 1.6.6, one concludes:

Proposition 3.2.1. *If $d > 2$, then the following conditions are equivalent:*

1. *The homology group $H_{d-2}(E_d(\ell); \mathbf{Z}_2)$ is non-trivial;*
2. *The number $a_{n-3}(\ell)$ is equal to one;*
3. *The length vector ℓ is generic and there is a diffeomorphism*

$$E_d(\ell) \simeq (S^{d-1})^{n-3} \times T^1 S^{d-1}.$$

Assume now that $a_{n-3}(\ell) = 0$. It follows from Proposition 1.6.10 that in this case the integral homology group $H_{d-2}(E_d(\ell); \mathbf{Z})$ is trivial. Using Theorem 3.1.1, the second part of Theorem 3.1.2 and the universal coefficient theorem, we conclude that for $d > 2$ the homology group $H_{d-1}(E_d(\ell); \mathbf{Z})$ is free abelian of rank $a_1(\ell) + b_1(\ell) + a_0(\ell) + b_0(\ell)$. By the first part of Proposition 1.5.1, $a_0(\ell) + b_0(\ell) = 1$ if the space $E_d(\ell)$ is non-empty. Combining Proposition 1.6.10 with the Hurewicz theorem, we find:

Proposition 3.2.2. *Let $d > 2$. If $a_{n-3}(\ell) = 0$ and $E_d(\ell) \neq \emptyset$, then the homotopy group $\pi_{d-1}(E_d(\ell))$ is free abelian of rank*

$$a_1(\ell) + b_1(\ell) + 1.$$

Proposition 3.2.2 becomes false for $d = 2$. For example, consider the case $n = 5$, $\ell = (1, 1, 1, 1, 1)$. The condition $a_2(\ell) = 0$ is satisfied. On the other hand, it follows from the second part of Proposition 1.5.2 and Theorem 2 in [27] that there are homeomorphisms

$$E_2(\ell) \simeq S^1 \times M_\ell \simeq S^1 \times \Sigma_4,$$

where Σ_4 is the closed oriented surface of genus four. Thus in this case the fundamental group $\pi_1(E_2(\ell))$ is not free abelian.

It is known that for non-generic length vectors ℓ the planar polygon spaces M_ℓ are manifolds with finitely many singular points corresponding to median subsets $J \subset \{1, \ldots, n\}$ ([7], Theorem 1.6). Let us use Theorem 3.1.1 to study the question whether the space $E_d(\ell)$ can be a manifold if ℓ is non-generic.

By Theorem 3.1.1, the greatest possible non-vanishing $\mathbf{Z}_2$-Betti number of the space $E_d(\ell)$ is

$$\dim_{\mathbf{Z}_2} H_{(d-1)(n-1)-1}(E_d(\ell); \mathbf{Z}_2) = a_0(\ell).$$

The number $a_0(\ell)$ is either zero (if the one-element set $\{m\}$ consisting of the index of a maximal entry of ℓ is long or median with respect to ℓ) or one (if $\{m\}$ is short with respect to ℓ). If $a_0(\ell) = 0$, then either $E_d(\ell) = \emptyset$ or there is a homeomorphism $E_d(\ell) \simeq S^{d-1}$. On the other hand, Theorem 3.1.1 shows that in the case $d > 2$ the space $E_d(\ell)$ fails to satisfy Poincaré duality for every non-generic length vector ℓ with $a_0(\ell) = 1$. Indeed, from the formulae for the $\mathbf{Z}_2$-Betti numbers we see that the condition

$$\dim_{\mathbf{Z}_2} H_{(d-1)k}(E_d(\ell); \mathbf{Z}_2) = \dim_{\mathbf{Z}_2} H_{(d-1)(n-k-1)-1}(E_d(\ell); \mathbf{Z}_2)$$

is satisfied for all $k = 1, \ldots, n-2$ if and only if all the numbers $b_k(\ell)$ vanish. We conclude:

Proposition 3.2.3. *For every non-generic length vector ℓ with $a_0(\ell) = 1$ the space $E_d(\ell)$ is not a topological manifold.*

Let us now apply Theorem 3.1.1 to compute the Euler characteristic of the space $E_d(\ell)$. Recall from Section 1.3 that $\mathcal{M}(\ell)$ denotes the set of all subsets $J \subset \{1, \ldots, n\}$ that are median with respect to the length vector ℓ.

Proposition 3.2.4. *The Euler characteristic of the space $E_d(\ell)$ is given by*

$$\chi(E_d(\ell)) = \begin{cases} |\mathcal{M}(\ell)| & \text{if } d \text{ is odd,} \\ 0 & \text{if } d \text{ is even.} \end{cases}$$

In particular, for every generic length vector ℓ the Euler characteristic of the space $E_d(\ell)$ is zero.

Proof. The Euler characteristic of $E_2(\ell)$ vanishes since this space is homeomorphic to the product $S^1 \times M_\ell$ by the first part of Proposition 1.5.2.

Suppose that $d > 2$ is even. Applying Theorem 3.1.1,

$$\chi(E_d(\ell)) = \sum_{k=0}^{n-2}(-1)^k(a_k(\ell) + b_k(\ell) + a_{k-1}(\ell) + b_{k-1}(\ell)) + \sum_{k=1}^{n-1}(-1)^{k-1}(a_{n-1-k}(\ell) + a_{n-2-k}(\ell)).$$

Since $a_{-1}(\ell) = b_{-1}(\ell) = 0$ and $a_{n-2}(\ell) = b_{n-2}(\ell) = 0$, both sums vanish and thus $\chi(E_d(\ell)) = 0$.

In the case where d is odd, one computes

$$\begin{aligned}\chi(E_d(\ell)) &= \sum_{k=0}^{n-2}(a_k(\ell) + b_k(\ell) + a_{k-1}(\ell) + b_{k-1}(\ell)) \\ &\quad - \sum_{k=1}^{n-1}(a_{n-1-k}(\ell) + a_{n-2-k}(\ell)) \\ &= 2\sum_{k=0}^{n-3} b_k(\ell) = |\mathcal{M}(\ell)|.\end{aligned}$$

Here we used the fact that the number $b_{n-2}(\ell)$ is zero as every subset $J \subset \{1, \dots, n\}$ which contains the index of a maximal entry and has cardinality $|J| = n - 1$ must be long with respect to ℓ. The sum $\sum_{k=0}^{n-3} b_k(\ell)$ is the number of all median sets containing the index of a maximal entry and is thus equal to $\frac{1}{2}|\mathcal{M}(\ell)|$. □

Let us give a second, more direct proof of Proposition 3.2.4. The proof relies on the well-known fact that the Euler characteristic of a space equipped with a continuous S^1-action coincides with the Euler characteristic of the set of fixed points of the action (this is a consequence of the Lefschetz fixed point theorem; a proof can be found in [31], Theorem 5.5).

Fix a two-dimensional subspace $E \subset \mathbb{R}^d$ and an orientation of E. Consider the S^1-action on $\mathbb{R}^d$ where every element of S^1 acts by a linear map that rotates each vector in E in the positive direction and fixes every vector which is orthogonal to

E. We study the induced S^1-action on $E_d(\ell)$. If $d = 2$, then this action is free and thus

$$\chi(E_2(\ell)) = 0. \tag{3.1}$$

The set of fixed points of the S^1-action on $E_3(\ell)$ consists exactly of those polygons (viewed up to translations), where all the edges are perpendicular to E. Since every polygon of this type corresponds to a unique median subset of $\{1, \ldots, n\}$, the number of fixed points is $|\mathcal{M}(\ell)|$. Thus

$$\chi(E_3(\ell)) = |\mathcal{M}(\ell)|. \tag{3.2}$$

For $d > 3$, the set of fixed points of the action may be identified with $E_{d-2}(\ell)$ and thus

$$\chi(E_d(\ell)) = \chi(E_{d-2}(\ell)) \text{ for } d > 3. \tag{3.3}$$

Equations (3.1)-(3.3) imply that $\chi(E_d(\ell)) = \chi(E_2(\ell)) = 0$ if d is even and $\chi(E_d(\ell)) = \chi(E_3(\ell)) = |\mathcal{M}(\ell)|$ for d odd.

We now evaluate the $\mathbf{Z}_2$-Betti numbers in the cases of Propositions 1.6.3, 1.6.5 and 1.6.6.

Example 3.2.5. Consider the case of a massive edge (see Proposition 1.6.3): assume that the length vector ℓ is generic and that there is a maximal short one-element subset $\{m\} \subset \{1, \ldots, n\}$. In this case $a_0(\ell) = 1$ and $a_k(\ell) = 0$ for $k > 0$. The numbers $b_k(\ell)$ vanish. Thus by Theorem 3.1.1 the non-zero $\mathbf{Z}_2$-Betti numbers are

$$\dim_{\mathbf{Z}_2} H_0(E_d(\ell); \mathbf{Z}_2) = \dim_{\mathbf{Z}_2} H_{(d-1)(n-1)-1}(E_d(\ell); \mathbf{Z}_2) = 1$$

and

$$\dim_{\mathbf{Z}_2} H_{d-1}(E_d(\ell); \mathbf{Z}_2) = \dim_{\mathbf{Z}_2} H_{(d-1)(n-2)-1}(E_d(\ell); \mathbf{Z}_2) = 1.$$

This can be checked using Proposition 1.6.3. Namely, for $n > 3$ there is a diffeomorphism

$$E_d(\ell) \simeq S^{(d-1)(n-2)-1} \times S^{d-1}$$

and the Betti numbers are as indicated above.

If $n = 3$, then by Proposition 1.6.3,

$$E_d(\ell) \simeq T^1 S^{d-1}.$$

The homology groups of the latter space are easily computed from the Gysin exact sequence (see e.g. Section 4.D in [19])

$$\cdots \to H_{p+(d-1)}(T^1 S^{d-1}; \mathbf{Z}) \to H_{p+(d-1)}(S^{d-1}; \mathbf{Z}) \to H_p(S^{d-1}; \mathbf{Z}) \to \dots$$

Here the map $H_{p+(d-1)}(S^{d-1}; \mathbf{Z}) \to H_p(S^{d-1}; \mathbf{Z})$ is the cap product with the Euler class of the tangent bundle of S^{d-1}; this class is zero if d is even and is given by twice the generator of $H^{d-1}(S^{d-1}; \mathbf{Z})$ if d is odd. Thus

$$H_{d-1}(T^1 S^{d-1}; \mathbf{Z}) \simeq \begin{cases} 0 & \text{if } d \text{ is odd,} \\ \mathbf{Z} & \text{if } d \text{ is even.} \end{cases} \tag{3.4}$$

and

$$H_{d-2}(T^1 S^{d-1}; \mathbf{Z}) \simeq \begin{cases} \mathbf{Z}_2 & \text{if } d \text{ is odd,} \\ \mathbf{Z} & \text{if } d \text{ is even.} \end{cases} \tag{3.5}$$

Applying the universal coefficient theorem,

$$H_{d-2}(T^1 S^{d-1}; \mathbf{Z}_2) \simeq H_{d-1}(T^1 S^{d-1}; \mathbf{Z}_2) \simeq \mathbf{Z}_2. \tag{3.6}$$

We see that for $n = 3$ the $\mathbf{Z}_2$-Betti numbers of the space $E_d(\ell)$ agree with the formulae of Theorem 3.1.1.

Example 3.2.6. Let us investigate how the $\mathbf{Z}_2$-Betti numbers of the spaces $E_d(\ell)$ change under the operation of inserting a small edge (see Proposition 1.6.5). Let ℓ be a generic length vector, $0 < \varepsilon < [\ell]$ and $\ell' = (\varepsilon, \ell)$. For a subset $J \subset \{1, \dots, n\}$, denote by $\widehat{J}$ the set $\widehat{J} = \{j + 1 : j \in J\}$. Then

$$J \in \mathcal{L}(\ell) \iff \widehat{J} \in \mathcal{L}(\ell') \iff \widehat{J} \cup \{1\} \in \mathcal{L}(\ell').$$

We conclude that

$$a_k(\ell') = a_{k-1}(\ell) + a_k(\ell)$$

for $k = 1, \dots, n-1$. The numbers $b_k(\ell)$ and $b_k(\ell')$ are all zero as ℓ and ℓ' are generic.

Using the Künneth theorem, we see that the formulae of Theorem 3.1.1 are consistent with the diffeomorphism

$$E_d(\ell') \simeq S^{d-1} \times E_d(\ell)$$

established in Proposition 1.6.5.

Example 3.2.7. We now consider the case of a massive triangle (see Proposition 1.6.6). Using Proposition 1.5.3, we can assume without loss of generality that ℓ is ordered. In this case existence of a massive triangle means that the two-element subset $\{n-2, n-1\} \subset \{1, \ldots, n\}$ is long with respect to ℓ.

Under the above assumptions, $a_k(\ell)$ is the number of subsets $J \subset \{1, \ldots, n\}$ so that $n \in J$, $n-2 \notin J$, $n-1 \notin J$ and $|J| = k+1$. We conclude that

$$a_k(\ell) = \binom{n-3}{k}.$$

The numbers $b_k(\ell)$ vanish. Using Theorem 3.1.1, we find that

$$\dim_{\mathbf{Z}_2} H_{(d-1)k}(E_d(\ell); \mathbf{Z}_2) = \binom{n-2}{k} \tag{3.7}$$

and

$$\dim_{\mathbf{Z}_2} H_{(d-1)k-1}(E_d(\ell); \mathbf{Z}_2) = \binom{n-2}{k-1} \tag{3.8}$$

for $k = 1, \ldots, n-2$. On the other hand, by Proposition 1.6.6,

$$E_d(\ell) \simeq (S^{d-1})^{n-3} \times T^1 S^{d-1}.$$

Computing the $\mathbf{Z}_2$-homology groups of this latter space with the help of the Künneth theorem and (3.6), one confirms formulae (3.7) and (3.8).

We now verify the criterion of Theorem 3.1.5 in the cases of Examples 3.2.5, 3.2.6 and 3.2.7.

Assume that there is a massive edge (see Proposition 1.6.3 and Example 3.2.5). We distinguish between the two cases $n = 3$ and $n > 3$. In the latter case there are no sets J as indicated in the second part of Theorem 3.1.5: the only short subset $J \subset \{1, \ldots, n\}$ with $m \in J$ is $J = \{m\}$ and every two-element set $\{i, j\}$ with

$i, j \neq m$ is short. Thus by Theorem 3.1.5 the homology groups $H_*(E_d(\ell); \mathbf{Z})$ are torsion-free. This agrees with the diffeomorphism

$$E_d(\ell) \simeq S^{(d-1)(n-2)-1} \times S^{d-1}$$

established in Proposition 1.6.3. If $n = 3$, then the two-element set $J = \{1, 2, 3\} - \{m\}$ obtained from the index set $\{1, 2, 3\}$ by removing the index m of any maximal entry of ℓ satisfies both conditions of the second part of Theorem 3.1.5. Thus if $n = 3$ and d is odd, then the homology group $H_{d-2}(E_d(\ell); \mathbf{Z})$ must contain torsion elements. This can be checked as follows: by Proposition 1.6.6,

$$E_d(\ell) \simeq T^1 S^{d-1}.$$

If d is odd, then $H_{d-2}(T^1 S^{d-1}; \mathbf{Z}) \simeq \mathbf{Z}_2$ by (3.5).

Consider now the operation of inserting a small edge (see Proposition 1.6.5 and Example 3.2.6). Let ℓ be generic and let $0 < \varepsilon < [\ell]$. The diffeomorphism

$$E_d(\varepsilon, \ell) \simeq S^{d-1} \times E_d(\ell)$$

established in Proposition 1.6.5 implies that the integral homology groups of the space $E_d(\varepsilon, \ell)$ contain torsion elements if and only if so do the integral homology groups of the space $E_d(\ell)$. This is consistent with Theorem 3.1.5, since the existence of subsets J as described in part two of this Theorem is simultaneous for ℓ and for $\ell' = (\varepsilon, \ell)$.

Finally, we verify the statement of Theorem 3.1.5 in the case of a massive triangle. Assume that ℓ is ordered and $a_{n-3}(\ell) = 1$ (see Proposition 1.6.6). Then the two-element set $\{n-2, n-1\}$ is long with respect to ℓ. It follows that every subset $J \subset \{1, \ldots n\}$ with $|J| = k + 1$, $n - 2 \in J$, $n - 1 \in J$ and $n \notin J$ satisfies both assumptions of the second part of Theorem 3.1.5: J is long and the set

$$I = J - \{n-2, n-1\} \cup \{n\}$$

is short. Applying Theorem 3.1.5, we conclude that if d is odd, then every homology group $H_{(d-1)k-1}(E_d(\ell); \mathbf{Z})$, $1 \leq k \leq n-2$ contains torsion elements. This can be confirmed using the diffeomorphism

$$E_d(\ell) \simeq (S^{d-1})^{n-3} \times T^1 S^{d-1},$$

the Künneth Theorem and (3.5).

3.3 The equilateral Case

In this section we consider the important special case of equilateral polygons. Homology groups of spaces of equilateral planar polygons were studied in [25] and in [26], see also [15]. For $d > 2$ we obtain the following result:

Proposition 3.3.1. *Let $d > 2$ and assume that $l_i = l_j$ for all $i, j = 1, \ldots, n$. Denote by $r = \lfloor \frac{n}{2} \rfloor$ the largest integer with $2r \leq n$.*

1. *The non-vanishing homology groups $H_p(E_d(\ell); \mathbf{Z})$ with $p \neq (d-1)r-1, (d-1)r$ are free abelian and concentrated in dimensions*

$$p = (d-1)k,\ 0 \leq 2k \leq n-2$$

and

$$p = (d-1)k - 1,\ n < 2k \leq 2n-2.$$

Their ranks are

$$rk\ H_{(d-1)k}(E_d(\ell); \mathbf{Z}) = \binom{n}{k},\ 0 \leq 2k \leq n-2$$

and

$$rk\ H_{(d-1)k-1}(E_d(\ell); \mathbf{Z}) = \binom{n}{k+1},\ n < 2k \leq 2n-2.$$

2. *If d is even, then the groups $H_{(d-1)r-1}(E_d(\ell); \mathbf{Z})$ and $H_{(d-1)r}(E_d(\ell); \mathbf{Z})$ are free abelian and their ranks are as follows. If $n = 2r+1$ is odd, then*

$$rk\ H_{(d-1)r-1}(E_d(\ell); \mathbf{Z}) = rk\ H_{(d-1)r}(E_d(\ell); \mathbf{Z}) = \binom{2r}{r+1}.$$

If $n = 2r$ is even, then

$$rk\ H_{(d-1)r-1}(E_d(\ell); \mathbf{Z}) = \binom{2r-1}{r+1}$$

and

$$rk\ H_{(d-1)r}(E_d(\ell);\mathbf{Z}) = \binom{2r-1}{r}.$$

Proof. In the equilateral case a subset $J \subset \{1,\dots,n\}$ is short (respectively long or median) with respect to ℓ if and only if its cardinality satisfies $2|J| < n$ (respectively $2|J| > n$ or $2|J| = n$). This allows to express the numbers $a_k(\ell)$ and $b_k(\ell)$ as binomial coefficients:

$$a_k(\ell) = \begin{cases} \binom{n-1}{k} & \text{if } 2(k+1) < n, \\ 0 & \text{if } 2(k+1) \geq n \end{cases}$$

and

$$b_k(\ell) = \begin{cases} \binom{n-1}{k} & \text{if } 2(k+1) = n, \\ 0 & \text{if } 2(k+1) \neq n. \end{cases}$$

Every subset $J \subset \{1,\dots,n\}$ satisfying the conditions of the second part of Theorem 3.1.5 must have cardinality $|J| = r$. Thus by Theorems 3.1.2 and 3.1.5 all groups $H_p(E_d(\ell);\mathbf{Z})$ except for possibly $H_{(d-1)r-1}(E_d(\ell);\mathbf{Z})$ are torsion-free. Applying the universal coefficient theorem for homology, one finds that for all $d > 2$ the Betti numbers $b_p(E_d(\ell))$ with $p \neq (d-1)r, (d-1)r-1$ coincide with the $\mathbf{Z}_2$-Betti numbers computed in Theorem 3.1.1. Substituting the above expressions for $a_k(\ell)$ and $b_k(\ell)$ into the formulae of Theorem 3.1.1, we obtain the first assertion.

If d is even, then by Theorem 3.1.3 the remaining group $H_{(d-1)r-1}(E_d(\ell);\mathbf{Z})$ is free abelian as well. Thus in this case

$$b_{(d-1)r}(E_d(\ell)) = \dim_{\mathbf{Z}_2} H_{(d-1)r}(E_d(\ell);\mathbf{Z}_2)$$

and

$$b_{(d-1)r-1}(E_d(\ell)) = \dim_{\mathbf{Z}_2} H_{(d-1)r-1}(E_d(\ell);\mathbf{Z}_2)$$

by the universal coefficient theorem. The second assertion now follows from the formulae of Theorem 3.1.1. □

Proposition 3.3.1 computes the integral homology groups of the spaces $E_d(\ell)$ in the equilateral case with the exception of the groups

$$H_{(d-1)r}(E_d(\ell);\mathbf{Z}) \text{ and } H_{(d-1)r-1}(E_d(\ell);\mathbf{Z}),$$

where $r = \lfloor \frac{n}{2} \rfloor$ and d is odd.

3.4 The Topology of random Polygons

Due to the possible existence of torsion elements in the groups $H_*(E_d(\ell); \mathbf{Z})$ when d is odd, the Betti numbers $b_p(E_d(\ell)) = \text{rk } H_p(E_d(\ell); \mathbf{Z})$ are in general different from the $\mathbf{Z}_2$-Betti numbers computed in Theorem 3.1.1. As we demonstrate in this section, the problem of computing the integral homology groups of the spaces $E_d(\ell)$ simplifies considerably in the setting of random linkages. Remarkably, the asymptotic values of the expectations of the Betti numbers can be computed explicitly. Before stating this result, we recall the main idea of the study of random linkages. We refer to [8] for a more detailed exposition.

It follows from Definition 1.1.1 that for every $\tau > 0$, the space $E_d(\tau\ell)$ obtained by rescaling the entries of the length vector $\ell \in \mathbb{R}^n_{>0}$ by the parameter τ can be identified with $E_d(\ell)$. Thus we can also parametrize the space of polygons by elements of the open unit simplex

$$\Delta^{n-1} = \{(l_1, \dots, l_n) \in \mathbb{R}^n_{>0} : \sum_{j=1}^{n} l_j = 1\}.$$

The heuristic idea that in applications polygons with different edge lengths may be encountered with different probability can now be made precise as follows.

We view ℓ as a random variable characterized by a probability measure ν_n on the unit simplex Δ^{n-1}. Topological invariants of the spaces $E_d(\ell)$ are now random functions and we are interested in their asymptotic behaviour as the number n of edges becomes large. In this section, the asymptotic behaviour for large n of the homotopy groups and of the expected Betti numbers of the spaces $E_d(\ell)$ is determined in the simplest case where ν_n is the normalized Lebesgue measure on Δ^{n-1}, that is when

$$\nu_n(A) = \frac{vol(A)}{vol(\Delta^{n-1})}$$

for every measurable subset $A \subset \Delta^{n-1}$. Here vol denotes the Lebesgue measure on Δ^{n-1}.

We first investigate the asymptotic behaviour of the homotopy groups.

Theorem 3.4.1. *Let ν_n be the normalized Lebesgue measure on Δ^{n-1} and let $d \geq 2$. For every $p \geq 0$, there are constants $C > 0$ and $a \in (0,1)$, so that for all $n \geq 3$ the homomorphism*

$$\pi_p(E_d(\ell)) \to \pi_p((S^{d-1})^n) \simeq \bigoplus_n \pi_p(S^{d-1})$$

induced by inclusion is an isomorphism with probability at least $1 - Ca^n$.

Next, we study the expectations of the Betti numbers. For every $p \geq 0$, denote by $E(b_p(E_d(\ell)))$ the mathematical expectation

$$E(b_p(E_d(\ell))) = \int_{\Delta^{n-1}} b_p(E_d(\ell)) d\nu_n$$

of the Betti number

$$b_p(E_d(\ell)) = \text{rk } H_p(E_d(\ell); \mathbf{Z})$$

of dimension p. It follows from the first part of Theorem 3.1.2 that

$$E(b_p(E_d(\ell))) = 0 \text{ for } p \neq 0, -1 \bmod (d-1).$$

Theorem 3.4.2. *Let ν_n be the normalized Lebesgue measure on the unit simplex Δ^{n-1}. Let $d > 2$.*

1. *For every $k \geq 0$ there exist constants $C > 0$ and $a \in (0,1)$, so that for all $n \geq 3$,*
$$\left| \int_{\Delta^{n-1}} b_{(d-1)k}(E_d(\ell)) d\nu_n - \binom{n}{k} \right| < Ca^n$$
2. *For every $k \geq 1$ there exist $C > 0$ and $a \in (0,1)$, so that for all $n \geq 3$,*
$$\left| \int_{\Delta^{n-1}} b_{(d-1)k-1}(E_d(\ell)) d\nu_n \right| < Ca^n.$$

Theorem 3.4.2 complements similar results for the expected Betti numbers of the spaces M_ℓ and N_ℓ obtained in [8]. Its proof is somewhat different: since we do not have explicit formulae for the Betti numbers of the spaces $E_d(\ell)$ in the case where d is odd, our argument is based on the result of Proposition 1.6.8.

The claims of Theorems 3.4.1 and 3.4.2 in fact hold for a large class of probability measures on the unit simplex, as described in Definition 1 of [8]. However, for simplicity the exposition here is restricted to the case of the normalized Lebesgue measure.

Before giving the proofs of Theorems 3.4.1 and 3.4.2, we introduce some notation. Recall that a subset $J \subset \{1, \ldots, n\}$ is called short with respect to the length vector ℓ if

$$\ell_J = \sum_{j \in J} l_j - \sum_{j \notin J} l_j < 0.$$

For every $k \geq 0$, we consider the subset $\Gamma_k \subset \Delta^{n-1}$ consisting of those length vectors ℓ, which have the property that every subset $J \subset \{1, \ldots, n\}$ of cardinality $|J| = k$ is short with respect to ℓ. The proofs of Theorems 3.4.1 and 3.4.2 use the following estimate of the volume of the subset Γ_k.

Lemma 3.4.3 ([8])**.** *For $1 \leq k \leq n$, denote by $\Gamma_k \subset \Delta^{n-1}$ the subset*

$$\Gamma_k = \{\ell \in \Delta^{n-1} : \ell_J < 0 \textit{ for all } J \subset \{1, \ldots, n\} \textit{ with } |J| = k\}.$$

Then

$$1 - n^{2k} 2^{-n} \leq \frac{vol(\Gamma_k)}{vol(\Delta^{n-1})} \leq 1.$$

The proof can be found in [8], Proposition 3 and we only recall the main idea. For every $J \subset \{1, \ldots, n\}$, consider the subset $V_J \subset \Delta^{n-1}$ given by $V_J = \{\ell \in \Delta^{n-1} : \ell_J \geq 0\}$. We can express $\Gamma_k \subset \Delta_{n-1}$ as the union

$$\Gamma_k = \Delta^{n-1} - \cup_{|J|=k} V_J.$$

One observes that $V_J \subset \Delta^{n-1}$ is the frustum obtained by intersecting $\Delta_{n-1} \subset \mathbb{R}^n$ with the half-space $\{\ell_J \geq 0\} \subset \mathbb{R}^n$ and proceeds by applying known explicit formulae for the volume of a frustum of a simplex.

We are ready to give the proof of Theorem 3.4.1.

Proof of Theorem 3.4.1. If $\ell \in \Gamma_k$, $k \geq 1$, then every subset $J \subset \{1, \dots, n\}$ which is long with respect to ℓ has cardinality $|J| > k$. It follows from Proposition 1.6.8 that in this case the inclusion homomorphism

$$i_p : \pi_p(E_d(\ell)) \to \pi_p((S^{d-1})^n)$$

is an isomorphism if $p < (d-1)k - 1$. Since $d \geq 2$, the probability for i_p to be an isomorphism can be estimated from below by

$$\nu_n(\Gamma_{p+2}) = \frac{vol(\Gamma_{p+2})}{vol(\Delta^{n-1})} \geq 1 - n^{2(p+2)} 2^{-n} > 1 - Ca^n,$$

where $a \in (1/2, 1)$ and where the constant $C > 0$ is chosen so that $C > n^{2(p+2)}(2a)^{-n}$ for all $n \geq 3$. $\square$

Proof of Theorem 3.4.2. Let $k \geq 0$ and let $\ell \in \Gamma_{k+1}$. By the proof of Theorem 3.4.1, the inclusion homomorphism $i_p : \pi_p(E_d(\ell)) \to \pi_p((S^{d-1})^n)$ is an isomorphism if

$$p < (d-1)(k+1) - 1.$$

It follows that for $\ell \in \Gamma_{k+1}$ the non-vanishing Betti numbers $b_p(E_d(\ell))$ with $p < (d-1)(k+1) - 1$ are

$$b_{(d-1)k'}(E_d(\ell)) = \binom{n}{k'},\ 0 \leq k' \leq k.$$

Next, we combine Theorem 3.1.1 and the universal coefficient Theorem to estimate the Betti numbers in the case where $\ell \in \Delta^{n-1} - \Gamma_{k+1}$. Consider the short exact sequence

$$0 \to H_p(E_d(\ell); \mathbf{Z}) \otimes \mathbf{Z}_2 \to H_p(E_d(\ell); \mathbf{Z}_2) \to \mathrm{Tor}(H_{p-1}(E_d(\ell); \mathbf{Z}), \mathbf{Z}_2) \to 0$$

of the universal coefficient theorem for homology. Using Theorem 3.1.1, the estimates

$$a_k(\ell) \leq \binom{n-1}{k} \text{ and } b_k(\ell) \leq \binom{n-1}{k}$$

and the above the exact sequence, one obtains

$$b_{(d-1)k}(E_d(\ell)) \leq \dim H_{(d-1)k}(E_d(\ell); \mathbf{Z}_2)$$
$$\leq 2\binom{n-1}{k} + 2\binom{n-1}{k-1} = 2\binom{n}{k} \leq 2n^k.$$

and similarly

$$\begin{aligned} b_{(d-1)k-1}(E_d(\ell)) &\leq \dim H_{(d-1)k-1}(E_d(\ell); \mathbf{Z}_2) \\ &\leq \binom{n-1}{n-1-k} + \binom{n-1}{n-2-k} \leq n^{k+1}. \end{aligned}$$

We can now compute

$$\begin{aligned} &\left| \int_{\Delta^{n-1}} b_{(d-1)k}(E_d(\ell)) d\nu_n - \binom{n}{k} \right| \\ = &\left| \int_{\Delta^{n-1}-\Gamma_{k+1}} b_{(d-1)k}(E_d(\ell)) d\nu_n - \left(1 - \frac{vol(\Gamma_{k+1})}{vol(\Delta^{n-1})}\right) \binom{n}{k} \right| \\ &\leq 2\left(1 - \frac{vol(\Gamma_{k+1})}{vol(\Delta^{n-1})}\right) n^k \leq n^{3k+2} 2^{-n+1} < Ca^n, \end{aligned}$$

where $a \in (1/2, 1)$ and the constant C is chosen so that $C > 2n^{3k+2}(2a)^{-n}$ for all $n \geq 3$. We have established the first claim the theorem. To prove the second claim, we compute, for $k \geq 1$,

$$\begin{aligned} \left| \int_{\Delta^{n-1}} b_{(d-1)k-1}(E_d(\ell)) d\nu_n \right| &= \left| \int_{\Delta^{n-1}-\Gamma_{k+1}} b_{(d-1)k-1}(E_d(\ell)) d\nu_n \right| \\ &\leq \left(1 - \frac{vol(\Gamma_{k+1})}{vol(\Delta^{n-1})}\right) n^{k+1} \leq n^{3k+3} 2^{-n} < Ca^n, \end{aligned}$$

where $a \in (1/2, 1)$ and C is chosen so that $C > n^{3k+3}(2a)^{-n}$ for all $n \geq 3$. This completes the proof. □

3.5 Proof of Theorem 3.1.1

It will be convenient to assume for the proof that the length vector ℓ is ordered (see section 1.3). Since every length vector can be obtained from an ordered length vector by a permutation of the entries and since by Proposition 1.5.3, the homeomorphism type of the space $E_d(\ell)$ does not change when we permute the entries of ℓ, this assumption is not restrictive.

Recall from Section 1.1 that the space $E_d(\ell)$ is a subset of $W = (S^{d-1})^n$. Our goal is to compute the dimensions of the kernel and of the cokernel of the homomorphism

$$j_k : H_{(d-1)k}(W - E_d(\ell); \mathbf{Z}_2) \to H_{(d-1)k}(W; \mathbf{Z}_2) \tag{3.9}$$

induced by the inclusion $W - E_d(\ell) \hookrightarrow W$.

Consider the robot arm distance map $f_\ell : W \to \mathbb{R}$,

$$(u_1, \dots, u_n) \mapsto -\Big| \sum_{j=1}^{n} l_j u_j \Big|^2.$$

By Lemma 1.6.1, the critical points of f_ℓ consist of the zero level set $f_l^{-1}(0) = E_d(\ell)$ and all the submanifolds

$$P_J = \{(u_1, \dots, u_n) : \ u_i = u_j = -u_k \text{ for } i, j \in J, k \notin J\},$$

where $J \subset \{1, \dots, n\}$ is long with respect to ℓ. The Morse-Bott index of P_J is $\mathrm{ind}_{f_l}(P_J) = (d-1)(n - |J|)$.

Fix $\varepsilon \in (0, [\ell])$ and consider the subset

$$W^\varepsilon = f_\ell^{-1}((-\infty, -\varepsilon^2)) \subset W.$$

Since $-\varepsilon^2$ is a regular value of f_ℓ, W^ε is a manifold with boundary $\partial W^\varepsilon = f_\ell^{-1}(-\varepsilon^2)$. The flow of f_ℓ defines a retraction of $f_\ell^{-1}(-\infty, 0) = W - E_d(\ell)$ onto W^ε.

Due to the choice of ε, for every subset $J \subset \{1, \dots, n\}$ which is long with respect to ℓ there is an inclusion $P_J \subset W^\varepsilon$. Every submanifold P_J is diffeomorphic to the sphere S^{d-1} and its Morse-Bott index is a multiple of $(d-1)$. Thus if $d > 2$, then the pair (W^ε, f_ℓ) satisfies the assumptions of Theorem 2.2.2. It follows that in this case the restriction of f_ℓ to the complement $W - E_d(\ell)$ is perfect and hence

$$H_*(W - E_d(\ell); \mathbf{Z}_2) \simeq H_*(W^\varepsilon; \mathbf{Z}_2) \simeq \bigoplus_{J \text{ long w.r.t. } \ell} H_{*-\mathrm{ind}_{f_l}(P_J)}(P_J; \mathbf{Z}_2).$$

Next, we use the criterion of Theorem 2.3.1 to identify a homology basis for $W - E_d(\ell)$.

Fix $e \in S^{d-1}$ and define for every subset $J \subset \{1, \dots, n\}$ submanifolds $V_J, W_J \subset W$ by

$$V_J = \{(u_1, \dots, u_n) \in W : \ u_j = e \text{ for } j \in J\} \tag{3.10}$$

and

$$W_J = \{(u_1, \ldots, u_n) \in W : \; u_i = u_j \text{ for } i, j \in J\}. \tag{3.11}$$

Thus V_J is obtained by fixing the direction of all the segments of the robot arm whose indices lie in J to be $e \in S^{d-1}$ while W_J consists of all those configurations of the robot arm, where all the segments whose indices lie in J are parallel.

The dimensions of V_J and of W_J are

$$\dim V_J = (d-1)(n - |J|)$$

and

$$\dim W_J = (d-1)(n - |J| + 1).$$

For every $J \subset \{1, \ldots, n\}$, $V_J \subset W_J$. If the set J is long with respect to ℓ, then $W_J \subset W^\varepsilon$. Moreover in this case the class $[V_J] \in H_{(d-1)(n-|J|)}(W_\varepsilon; \mathbf{Z}_2)$ is independent of the choice of the point $e \in S^{d-1}$.

We claim that the assumptions of Theorem 2.3.1 are satisfied. To this end, note that the submanifolds P_J and V_J of W_J intersect transversally in a single point $e_J \in W$ given by

$$u_j = \begin{cases} e & \text{if } j \in J, \\ -e & \text{if } j \notin J. \end{cases}$$

Moreover, the inequality $f_\ell(q) > f_\ell(q')$ holds for all $q \in P_J$, $q' \in W_J - P_J$. Indeed, $W_J \subset W$ is the subset of those configurations of the robot arm, where all segments with indices in J point in the same direction. Since the function f_ℓ measures the negative distance between the initial point and the endpoint of the robot arm, the maximum of f_ℓ on W_J is attained exactly at those configurations of the arm, where all the segments with indices not in J point in the opposite direction.

Applying Theorem 2.3.1, we obtain:

Proposition 3.5.1. *The classes $[W_J], [V_K]$ where $J, K \subset \{1, \ldots, n\}$ are long with respect to ℓ and $|J| = n - k + 1$, $|K| = n - k$, form a basis of the $\mathbf{Z}_2$-vector space $H_{(d-1)k}(W - E_d(\ell); \mathbf{Z}_2)$.*

For $1 \leq k \leq n$, a basis of $H_{(d-1)k}(W; \mathbf{Z}_2)$ is given by the classes $[V_K]$, where $K \subset \{1, \ldots, n\}$ is a subset of cardinality $|K| = n - k$. For $|K| + |K'| = n$, the mod 2 intersection number of the classes $[V_K]$ and $[V_{K'}]$ is

$$[V_K] \cdot [V_{K'}] = \begin{cases} 1 & \text{if } K \cap K' = \emptyset, \\ 0 & \text{if } K \cap K' \neq \emptyset. \end{cases}$$

The intersection numbers involving the classes $[W_J]$ are as follows:

Lemma 3.5.2. *1. For $J, K \subset \{1, \ldots, n\}$, $|J| + |K| = n + 1$ the intersection number $[W_J] \cdot [V_K] \in \mathbf{Z}_2$ of the classes $[W_J]$ and $[V_K]$ is given by*

$$[W_J] \cdot [V_K] = \begin{cases} 1 & \textit{if } |J \cap K| = 1, \\ 0 & \textit{if } |J \cap K| > 1. \end{cases}$$

2. For every pair of subsets $I, J \subset \{1, \ldots, n\}$ with $|I| + |J| = n+2$ the intersection number $[W_I] \cdot [W_J] \in \mathbf{Z}_2$ vanishes.

Proof. Let $|J \cap K| > 1$. Fix $k \in K \cap J$ and $e' \in S^{d-1}$ with $e' \neq e$. Consider the submanifold $V'_K \subset W$ given by

$$V'_K = \{(u_1, \ldots, u_n) \in W : u_j = e \text{ for } j \in K - \{k\} \text{ and } u_k = e'\}.$$

Then $[V_K] = [V'_K]$ and $W_J \cap V'_K = \emptyset$. We conclude that in this case $[W_J] \cdot [V_K] = 0$.

If $|J \cap K| = 1$ then the submanifolds W_J and V_K have a unique point of intersection given by $u_j = e$ for $j = 1, \ldots, n$. This intersection is transverse.

To demonstrate the second part of the claim, let $j \in I \cap J$. Fix a diffeomorphism $\varphi : S^{d-1} \to S^{d-1}$ which is homotopic to the identity and has no fixed points (if d is even) or two nondegenerate fixed points (if d is odd). Define $\Phi : (S^{d-1})^n \to (S^{d-1})^n$ as the map which applies φ to the jth factor and fixes all other factors. Then $[\Phi(W_I)] = [W_I]$ and the submanifolds $\Phi(W_I)$ and W_J are either disjoint or intersect transversally in exactly two points. Thus the mod 2 intersection number is zero. □

Let us use Lemma 3.5.2 to choose a basis of $H_*(W; \mathbf{Z}_2)$ which will be more convenient for the proof of Theorem 3.1.1.

Lemma 3.5.3. *For $0 \leq k \leq n$, there is a basis of $H_{(d-1)k}(W; \mathbf{Z}_2)$ consisting of the classes $[W_J]$ and $[V_K]$ where $J \subset \{1, \ldots, n\}$ is a subset with $|J| = n - k + 1$ and $n \in J$ and $K \subset \{1, \ldots, n\}$ a subset with $|K| = n - k$ and $n \in K$.*

Proof. Denote $H_k = H_{(d-1)k}(W; \mathbf{Z}_2)$ and let $H'_k \subset H_k$ be the subspace generated by the classes $[W_J]$ and $[V_K]$ as in the claim of the Lemma. Since the number $\binom{n-1}{k-1} + \binom{n-1}{k}$ of the specified generators of H'_k coincides with the rank $\binom{n}{k}$ of H_k, it suffices to check that the intersection form of W restricts to a nondegenerate bilinear form

$$H'_k \times H_{n-k} \to \mathbf{Z}_2.$$

There is a basis of H_{n-k} consisting of the classes $[V_I]$ so that $I \subset \{1, \ldots, n\}$ and $|I| = k$. We denote by $E_{n-k} \subset H_{n-k}$ (respectively by $F_{n-k} \subset H_{n-k}$) the subspace generated by all the classes $[V_I]$ so that $|I| = k$ and $n \in I$ (respectively by all the classes $[V_I]$ so that $|I| = k$ and $n \notin I$). Further, we write $E'_k \subset H'_k$ (respectively $F'_k \subset H'_k$) for the subspace generated by all the classes $[W_J]$ with $|J| = n - k + 1$ and $n \in J$ (respectively by all the classes $[V_K]$ with $|K| = n - k$ and $n \in J$). Then

$$H'_k = E'_k \oplus F'_k$$

and

$$H_{n-k} = E_{n-k} \oplus F_{n-k}.$$

Since the intersection number $[V_K] \cdot [V_I]$ is zero if both sets K and I contain the index n, the intersection form vanishes identically on $F'_k \times E_{n-k}$. It follows from Lemma 3.5.2 that for $[W_J] \in E'_k$ and $[V_I] \in E_{n-k}$ the intersection number $[W_J] \cdot [V_I]$ is non-zero if and only if $I = \overline{J} \cup \{n\}$. Thus the intersection form is nondegenerate on $E'_k \times E_{n-k}$. One concludes analogously that the intersection form is nondegenerate on $F'_k \times F_{n-k}$. This completes the proof. □

Remark 3.5.4. It follows from Lemma 3.5.2 that the duals with respect to the intersection form on $H_*(W; \mathbf{Z}_2)$ of the basis elements of Lemma 3.5.3 are given by

$$[W_J]^* = [V_{\widehat{J}}] \text{ and } [V_K]^* = [W_{\widehat{K}}].$$

Here for a subset $J \subset \{1, \ldots, n\}$ with $n \in J$ the symbol $\widehat{J}$ denotes the set $\widehat{J} = \overline{J} \cup \{n\}$.

We are ready to determine the dimensions of the kernel and of the cokernel of homomorphism (3.9). We write $H_{(d-1)k}(W - E_d(\ell); \mathbf{Z}_2)$ as a direct sum

$$H_{(d-1)k}(W - E_d(\ell); \mathbf{Z}_2) = A_k \oplus A'_k \oplus B_k \oplus B'_k,$$

where:

- A_k (respectively A'_k) is generated by classes $[W_J]$ with $|J| = n - k + 1$, J long with respect to ℓ and $n \in J$ (respectively $n \notin J$).
- B_k (respectively B'_k) is generated by classes $[V_K]$ with $|K| = n - k$, K long with respect to ℓ and $n \in K$ (respectively $n \notin K$).

We write

$$H_{(d-1)k}(W; \mathbf{Z}_2) = A_k \oplus B_k \oplus C_k \oplus D_k,$$

where A_k and B_k are as above and

- C_k is generated by the classes $[W_J]$ with $|J| = n - k + 1$, $n \in J$ and J either short or median with respect to ℓ.
- D_k is generated by the classes $[V_K]$ with $|K| = n - k$, $n \in K$ and K either short or median with respect to ℓ.

The homomorphism j_k restricts to the identity map on $A_k \oplus B_k$.

Lemma 3.5.5. *There are inclusions*

$$j_k(A'_k) \subset A_k \tag{3.12}$$

and

$$j_k(B'_k) \subset A_k \oplus B_k. \tag{3.13}$$

Proof. Assume $[W_I] \in A'_k$ and thus $|I| = n - k + 1$, I is long with respect to ℓ and $n \notin I$. Using Remark 3.5.4, one obtains the following Fourier decomposition:

$$j_k([W_I]) = \sum_{[W_J]} ([W_I] \cdot [V_{\widehat{J}}])[W_J] + \sum_{[V_K]} ([W_I] \cdot [W_{\widehat{K}}])[V_K]. \tag{3.14}$$

Here the first sum is over subsets $J \subset \{1, \ldots, n\}$ with $|J| = n - k + 1$ and $n \in J$ and the second sum is over subsets $K \subset \{1, \ldots, n\}$ with $|K| = n - k$ and $n \in K$.

By Lemma 3.5.2, every coefficient $[W_I] \cdot [W_{\widehat{K}}]$ in the second sum on the right-hand side of (3.14) vanishes. Thus

$$j_k([W_I]) \in A_k \oplus C_k.$$

To show that the class $j_k([W_I])$ lies in A_k, we must check that for every subset $J \subset \{1, \ldots, n\}$ which is either short or median with respect to ℓ and so that $|J| = n - k + 1$ and $n \in J$, we have $[W_I] \cdot [V_{\widehat{J}}] = 0$. Recall that the intersection number $[W_I] \cdot [V_{\widehat{J}}]$ is non-zero if and only if the subsets I and $\widehat{J}$ have exactly one element in common. The latter condition is equivalent to the set J being of the form $J = I - \{i\} \cup \{n\}$ for some $i \in I$. Since l_n is a maximal entry of ℓ, this is a contradiction to the assumption that I is long and J is short or median with respect to ℓ. We have established (3.12).

We now demonstrate (3.13). Let $[V_L] \in B'_k$. This means that $|L| = n - k$, L is long with respect to ℓ and $n \notin L$. As above, there is a decomposition

$$j_k([V_L]) = \sum_{[W_J]} ([V_L] \cdot [V_{\widehat{J}}])[W_J] + \sum_{[V_K]} ([W_L] \cdot [W_{\widehat{K}}])[V_K], \tag{3.15}$$

where the sums are over subsets J and K of $\{1, \ldots, n\}$ containing the index n with $|J| = n - k + 1$ and $|K| = n - k$. We must show that in the two sums on the right-hand side of (3.15) all the coefficients corresponding to basis elements $[W_J]$, $[V_K]$ which lie in C_k respectively in D_k vanish.

If $[W_J] \in C_k$, then $n \in J$ and J is either short or median with respect to ℓ. The intersection number $[V_L] \cdot [V_{\widehat{J}}]$ is non-zero if and only if $J = L \cup \{n\}$, contradicting the assumption that L is long and J is short or median with respect to ℓ. Let $[V_K] \in D_k$. This means that $|K| = n - k$, $n \in K$ and that K is short or median with respect to ℓ. The intersection number $[V_J] \cdot [W_{\widehat{K}}]$ is non-zero only if $K = J - \{j\} \cup \{n\}$ for some $j \in J$. This contradicts the assumption that J is long and K is short or median with respect to ℓ. This completes the proof of (3.13). $\square$

Corollary 3.5.6. *The image of the homomorphism j_k is generated by all the classes $[W_J]$ and $[V_K]$ so that the subsets $J, K \subset \{1, \dots, n\}$ are long with respect to ℓ, contain the index n and have the cardinalities $|J| = n - k + 1$ and $|K| = n - k$.*

Thus

$$\dim_{\mathbf{Z}_2} \mathrm{coker}(j_k) = \dim_{\mathbf{Z}_2} C_k + \dim_{\mathbf{Z}_2} D_k$$

$$= a_{n-k}(\ell) + b_{n-k}(\ell) + a_{n-k-1}(\ell) + b_{n-k-1}(\ell).$$

On the other hand, from Proposition 3.5.5

$$\dim_{\mathbf{Z}_2} \ker(j_k) = \dim_{\mathbf{Z}_2} A'_k + \dim_{\mathbf{Z}_2} B'_k = a_{k-2}(\ell) + a_{k-1}(\ell).$$

Consider the exact sequence of the pair $(W, W - E_d(\ell))$ for homology with $\mathbf{Z}_2$-coefficients. Using the fact that the non-vanishing homology groups $H_*(W; \mathbf{Z}_2)$ and $H_*(W - E_d(\ell); \mathbf{Z}_2)$ are concentrated in dimensions which are multiples of $d - 1$, we see that non-vanishing homology groups $H_p(W, W - E_d(\ell); \mathbf{Z}_2)$ are concentrated in dimensions of the form $p = (d-1)k$ and $p = (d-1)k + 1$, $k \geq 0$. Moreover, if $d > 2$, then there is an exact sequence

$$0 \to H_{(d-1)k+1}(W, W - E_d(\ell); \mathbf{Z}_2) \to H_{(d-1)k}(W - E_d(\ell); \mathbf{Z}_2)$$

$$\xrightarrow{j_k} H_{(d-1)k}(W; \mathbf{Z}_2) \to H_{(d-1)k}(W, W - E_d(\ell); \mathbf{Z}_2) \to 0.$$

Using Poincaré-Lefschetz duality and excision,

$$H_*(W, W - E_d(\ell); \mathbf{Z}_2) \simeq H^{(d-1)n-*}(E_d(\ell); \mathbf{Z}_2).$$

Thus

$$\begin{aligned} \dim_{\mathbf{Z}_2} H_{(d-1)k}(E_d(\ell); \mathbf{Z}_2) &= \dim_{\mathbf{Z}_2} H^{(d-1)k}(E_d(\ell); \mathbf{Z}_2) \\ &= \dim_{\mathbf{Z}_2} H_{(d-1)(n-k)}(W, W - E_d(\ell); \mathbf{Z}_2) \\ &= \dim_{\mathbf{Z}_2} \mathrm{coker}(j_{n-k}) \\ &= a_k(\ell) + b_k(\ell) + a_{k-1}(\ell) + b_{k-1}(\ell) \end{aligned}$$

and

$$\begin{aligned}\dim_{\mathbf{Z}_2} H_{(d-1)k-1}(E_d(\ell);\mathbf{Z}_2) &= \dim_{\mathbf{Z}_2} H^{(d-1)k-1}(E_d(\ell);\mathbf{Z}_2)\\ &= \dim_{\mathbf{Z}_2} H_{(d-1)(n-k)+1}(W, W-E_d(\ell);\mathbf{Z}_2)\\ &= \dim_{\mathbf{Z}_2}\ker(j_{n-k})\\ &= a_{n-k-2}(\ell)+a_{n-k-1}(\ell).\end{aligned}$$

The proof of Theorem 3.1.1 is complete.

3.6 Proof of Theorems 3.1.2, 3.1.3 and 3.1.5

Proof of Theorem 3.1.2. In course of the proof of Theorem 3.1.1 in the previous section, homology classes in $H_*(W;\mathbf{Z}_2)$ defined by submanifolds $V_K, W_J \subset W$ were used. In order to obtain integral homology classes, we now orient the submanifolds V_K, W_J .

We orient each submanifold $V_K \subset W$ as the product $V_K = (S^{d-1})^{n-|K|} \subset (S^{d-1})^n = W$. In more detail, let $\{i_1 < \cdots < i_k\}$ be the complement of K in $\{1,\dots,n\}$. For every element $u=(u_1,\dots,u_n)\in V_K$, let $B_{i_1},\dots,B_{i_k}$ be bases of the tangent spaces $T_{u_{i_1}}S^{d-1},\dots,T_{u_{i_k}}S^{d-1}$ which are positive with respect to the standard orientation of the sphere. We fix an orientation of the tangent space T_uV_K by defining the basis $(B_{i_1},\dots,B_{i_k})$ to be positive.

Every submanifold $W_J \subset W$ is oriented as the product of the corresponding manifold V_J and the diagonal in $(S^{d-1})^{n-|J|}$. More precisely, let $u=(u_1,\dots,u_n)\in W_J$. By definition of W_J, this means that $u_j = e$ for all $j\in J$. We identify $T_uW_J \simeq T_uV_J \oplus T_eS^{d-1}$ and orient T_uW_J by declaring a basis of T_uW_J consisting of a positive basis of T_uV_J, followed by a positive basis of T_eS^{d-1}, to be positive.

We now apply Corollary 2.3.2 to conclude:

Proposition 3.6.1. *The homology classes* $[W_J],[V_K]$ *with* $|J|=n-k+1$, $|K| =$

$n-k$ and so that the sets J and K are long with respect to ℓ, form a free basis of $H_{(d-1)k}(W - E_d(\ell); \mathbf{Z})$.

Consider the exact sequence of the pair $(W, W - E_d(\ell))$ for homology with integral coefficients. The non-vanishing homology groups $H_*(W - E_d(\ell); \mathbf{Z})$ are concentrated in dimensions which are multiples of $d-1$. Using excision and Poincaré duality, there is an isomorphism

$$H_*(W, W - E_d(\ell); \mathbf{Z}) \simeq H^{(d-1)n-*}(E_d(\ell); \mathbf{Z}).$$

We see from the exact sequence that the non-vanishing groups $H^p(E_d(\ell); \mathbf{Z})$ lie in dimensions

$$p = (d-1)k,\ 0 \leq k \leq n-2 \text{ and } p = (d-1)k - 1,\ 1 \leq k \leq n-1.$$

Moreover, cohomology groups of the form $H^{(d-1)k-1}(E_d(\ell); \mathbf{Z})$ are torsion-free. The two assertions of the theorem now follow from the universal coefficient theorem for cohomology. □

We will conclude Theorems 3.1.3 and 3.1.5 from the following result:

Lemma 3.6.2. *Let*

$$j_k : H_{(d-1)k}(W - E_d(\ell); \mathbf{Z}) \to H_{(d-1)k}(W; \mathbf{Z})$$

be the homomorphism induced by the inclusion $W - E_{d_\ell} \hookrightarrow W$.

1. *If d is even, then the cokernel of the homomorphism j_k is torsion-free.*

2. *Let d be odd. The following conditions are equivalent:*

   (a) The cokernel of j_k contains torsion elements;

   (b) There exists a subset $J \subset \{1, \ldots, n-1\}$ of cardinality $|J| = n-k+1$ which is long with respect to ℓ and indices $i, j \in J$, so that the set $I = J - \{i, j\} \cup \{n\}$ is short or median with respect to ℓ.

We now show that Theorems 3.1.3 and 3.1.5 follow from Lemma 3.6.2.

Proof of Theorems 3.1.3 and 3.1.5. As in the proof of Theorem 3.1.1, it follows from the exact homology sequence of the pair $(W, W - E_d(\ell))$ that there is an isomorphism

$$H^{(d-1)k}(E_d(\ell); \mathbf{Z}) \simeq \mathrm{coker}(j_{n-k})$$

If $T_{(d-1)k-1} \subset H_{(d-1)k-1}(E_d(\ell); \mathbf{Z})$ denotes the subgroup consisting of torsion elements, then by the universal coefficient theorem for cohomology,

$$H^{(d-1)k}(E_d(\ell); \mathbf{Z}) \simeq H_{(d-1)k}(E_d(\ell); \mathbf{Z}) \oplus T_{(d-1)k-1}.$$

Here we used the fact that the homology group $H_{(d-1)k}(E_d(\ell); \mathbf{Z})$ is free abelian by Theorem 3.1.2.

We see that the homology group $H_{(d-1)k-1}(E_d(\ell); \mathbf{Z})$ contains torsion elements if and only if the cokernel of the homomorphism j_{n-k} contains torsion elements. The claims of Theorems 3.1.3 and 3.1.5 now follow from the first respectively the second part of Lemma 3.6.2. $\square$

Recall that by Proposition 3.6.1, the group $H_{(d-1)k}(W - E_d(\ell); \mathbf{Z})$ has a free basis consisting of all the classes $[V_K]$, $[W_J]$ so that K and J are long subsets with $|K| = n - k$ and $|J| = n - k + 1$. We write $H_{(d-1)k}(W - E_d(\ell); \mathbf{Z})$ as a direct sum

$$H_{(d-1)k}(W - E_d(\ell); \mathbf{Z}) = E_k \oplus E'_k \oplus B_k \oplus B'_k \tag{3.16}$$

of free abelian groups. Here B_k and B'_k are defined as in the proof of Proposition 3.1.1 in the previous section: $B_k \subset H_{(d-1)k}(W - E_d(\ell); \mathbf{Z})$ (respectively $B'_k \subset H_{(d-1)k}(W - E_d(\ell); \mathbf{Z})$) is the subgroup generated by all the classes $[V_K]$ so that $|K| = n - k$, K is long with respect to ℓ and $n \in K$ (respectively all the classes $[V_K]$ so that $|K| = n - k$, K is long with respect to ℓ and $n \notin K$).

Moreover, E_k and E'_k are defined as follows:

- E_k is the subgroup generated by all the classes $[W_J]$ so that $|J| = n - k + 1$, $n \in J$, J is long with respect to ℓ and so that the set $J - \{n\}$ is either short or median with respect to ℓ.

- E'_k is the subgroup generated by all the remaining basis elements $[W_J]$: the generators of E'_k are all the classes $[W_J] \in E'_k$ so that $|J| = n - k + 1$ and one of the following two conditions is satisfied:

    1. J does not contain the index n and is long with respect to ℓ;

    2. J contains n and the set $J - \{n\}$ is long with respect to ℓ.

There is a free basis of $H_{(d-1)k}(W; \mathbf{Z})$ consisting of the classes $[V_K]$, where $K \subset \{1, \ldots, n\}$ is a subset of cardinality $|K| = n - k$. We write $H_{(d-1)k}(W; \mathbf{Z})$ as a direct sum

$$H_{(d-1)k}(W; \mathbf{Z}) = F_k \oplus F'_k \oplus B_k \oplus B'_k \tag{3.17}$$

of free abelian groups, where the groups B_k and B'_k are defined as above and the groups F_k and F'_k are defined as follows:

- F_k is generated by all the classes $[V_K]$ with $|K| = n - k$, $n \notin K$ and so that the set K is short or median with respect to ℓ and the set $K \cup \{n\}$ is long.

- F'_k is generated by all the remaining basis elements $[V_K]$. Namely, generators of F'_k are all the classes $[V_K]$ so that $|K| = n - k$ and so that one of the following two conditions is satisfied:

    1. K contains the index n and is short or median with respect to ℓ;

    2. K does not contain n and the set $K \cup \{n\}$ is short or median with respect to ℓ.

The homomorphism $j_k : H_{(d-1)k}(W - E_d(\ell); \mathbf{Z}) \to H_{(d-1)k}(W; \mathbf{Z})$ is the identity on $B_k \oplus B'_k$. It follows that the cokernel of j_k is isomorphic to the cokernel of the composition of j_k with the projection

$$\pi : F_k \oplus F'_k \oplus B_k \oplus B'_k \to (F_k \oplus F'_k \oplus B_k \oplus B'_k)/(B_k \oplus B'_k)$$

$$= F_k \oplus F'_k.$$

Denote by

$$j'_k : H_{(d-1)k}(W - E_d(\ell); \mathbf{Z}) \to F_k \oplus F'_k$$

the composition $\pi \circ j_k$. Next, we describe the homomorphism j_k'.

By construction, j_k' vanishes identically on the subgroup

$$B_k \oplus B_k' \subset H_{(d-1)k}(W - E_d(\ell); \mathbf{Z}).$$

Consider an element $[W_J]$ of the specified basis of $E_k \subset H_{(d-1)k}(W - E_d(\ell); \mathbf{Z})$. We compute

$$j_k'([W_J]) = \sum_{[V_K] \in F_k} \frac{[W_J] \cdot [V_{\overline{K}}]}{[V_K] \cdot [V_{\overline{K}}]} [V_K] + \sum_{[V_I] \in F_k'} \frac{[W_J] \cdot [V_{\overline{I}}]}{[V_I] \cdot [V_{\overline{I}}]} [V_I], \tag{3.18}$$

where the first sum is over all basis elements $[V_K]$ of F_k and the second sum over all basis elements $[V_I]$ of F_k'. By definition of the subgroup E_k, J is a long subset with $n \in J$ so that the set $J - \{n\}$ is either short or median with respect to ℓ. On the other hand, the set K does not contain the index n. Moreover, K is short or median with respect to ℓ and the set $K \cup \{n\}$ is long with respect to ℓ.

It follows from the proof of Proposition 3.5.2 that the intersection number $[W_J] \cdot [V_{\overline{K}}]$ is non-zero if and only if the sets J and $\overline{K}$ have exactly one element in common; moreover in this case we have $[W_J] \cdot [V_{\overline{K}}] = \pm 1$ (the exact sign can be determined from the above choice of orientations of the submanifolds W_J, V_K, but this will not be used). Since $n \in J$ and $n \in \overline{K}$, the coefficient of $[V_K]$ in the first sum on the right-hand side of (3.18) is non-zero if and only if $K = J - \{n\}$; moreover in this case the coefficient equals ± 1.

We define for every basis element $[V_K]$ of F_k

$$Y_K = \frac{[W_{K'}] \cdot [V_{\overline{K}}]}{[V_K] \cdot [V_{\overline{K}}]} [V_K] + \sum_{[V_I] \in F_k'} \frac{[W_{K'}] \cdot [V_{\overline{I}}]}{[V_I] \cdot [V_{\overline{I}}]} [V_I], \tag{3.19}$$

where K' denotes the set $K' = K \cup \{n\}$.

Let $\widetilde{F}_k \subset F_k \oplus F_k'$ be the free abelian group generated by the classes Y_K. We have $F_k \oplus F_k' = \widetilde{F}_k \oplus F_k'$. From the definition of Y_K and the above discussion of equation (3.18) we see that $j_k'([W_K]) = Y_K$ for $[W_K] \in E_k$. Thus

$$j_k'(E_k) = \widetilde{F}_k$$

and moreover j'_k restricts to an isomorphism of E_k onto $\widetilde{F}_k$. It follows that if

$$j''_k : H_{(d-1)k}(W - E_d(\ell); \mathbf{Z}) \to F'_k$$

denotes the composition $\pi' \circ j'_k$ of j'_k and the projection

$$\pi' : \widetilde{F}_k \oplus F'_k \to (\widetilde{F}_k \oplus F'_k)/\widetilde{F}_k = F'_k,$$

then the cokernel of j'_k is isomorphic to the cokernel of j''_k.

We now describe the homomorphism j''_k. By construction, j''_k vanishes identically on the subgroup

$$E_k \oplus B_k \oplus B'_k \subset H_{(d-1)k}(W - E_d(\ell); \mathbf{Z}).$$

Next, let $[W_J]$ be an element of the specified basis of E'_k. Consider the right-hand side of (3.18). The intersection number $[W_J] \cdot [V_{\overline{I}}]$ is non-zero if and only if the sets J and $\overline{I}$ have exactly one element in common or, equivalently, if J is obtained from I by adding a single element of the complement $\overline{I}$. Examining the definitions of the subgroups E'_k and F'_k, we find that this is impossible. Indeed, assume that $n \in I$ and $J = I \cup \{j\}$ for some $j \in \overline{I}$. Since $n \in J$, it follows from the definition of E'_k that the set $J - \{n\} = I \cup \{j\} - \{n\}$ is long with respect to ℓ. However, since l_n is a maximal entry of ℓ, in this case the set I must be long as well in contradiction to the definition of F'_k. For $n \notin I$ we argue similarly: since in this case by the definition of the subgroup F'_k the set $I \cup \{n\}$ is short or median with respect to ℓ, so is any set of the form $I \cup \{j\}$, $j \in \overline{I}$.

It follows that for every basis element $[W_J] \in E'_k$, the second sum on the right-hand side of (3.18) vanishes. Thus

$$j'_k([W_J]) = \sum_{[V_K] \in F_k} \frac{[W_J] \cdot [V_{\overline{K}}]}{[V_K] \cdot [V_{\overline{K}}]} [V_K]$$

$$= \sum_{[V_K] \in F_k} \frac{[W_J] \cdot [V_{\overline{K}}]}{[W_{K'}] \cdot [V_{\overline{K}}]} (Y_K - \sum_{[V_I] \in F'_k} \frac{[W_{K'}] \cdot [V_{\overline{I}}]}{[V_I] \cdot [V_{\overline{I}}]} [V_I]).$$

Here Y_K is the homology class defined in (3.19).

One concludes that

$$j_k''([W_J]) = -\sum_{[V_K],[V_I]} \frac{[W_J]\cdot[V_{\overline{K}}]}{[W_{K'}]\cdot[V_{\overline{K}}]} \frac{[W_{K'}]\cdot[V_{\overline{I}}]}{[V_I]\cdot[V_{\overline{I}}]}[V_I], \tag{3.20}$$

where the sum is over all the elements $[V_K] \in F_k$ and $[V_I] \in F_k'$ of the specified bases of F_k and F_k'.

We will use (3.20) to analyze the cokernel of the homomorphism j_k''. The result of our analysis is summarised by the following Lemma:

Lemma 3.6.3. *Let $d > 2$.*

1. *Every element of the image of the homomorphism j_k'' is divisible by two.*
2. *If d is even, then the homomorphism j_k'' vanishes identically.*
3. *If d is odd, then the following conditions are equivalent:*
   - *(a) j_k'' is not the zero homomorphism;*
   - *(b) There exists a subset $J \subset \{1, \dots, n-1\}$ as in condition (b) in the second part of Lemma 3.6.2.*

Let us show that Lemma 3.6.2 follows from Lemma 3.6.3.

Proof of Lemma 3.6.2. By construction of the homomorphism j_k'', the cokernels of j_k and of j_k'' are isomorphic. Thus the first claim of Lemma 3.6.2 follows from the second assertion of Lemma 3.6.3.

We now demonstrate that if the homomorphism j_k'' does not vanish identically, then its cokernel contains torsion elements. Using the criterion of the third part of Lemma 3.6.3, this will imply the second assertion of Lemma 3.6.2.

Assume that $j_k''([W_J]) \neq 0$ for some basis element $[W_J] \in E_k'$. By the first part of Lemma 3.6.3, the element $j_k''([W_J]) \in F_k'$ is divisible by two. Let $s > 0$ be the maximal exponent so that 2^s divides $j_k''([W_J])$ and let

$$j_k''([W_J]) = 2^s X,\ X \in F_k'.$$

Then $2^s X \in \mathrm{Im}\ j_k''$ but $X \notin \mathrm{Im}\ j_k''$. Thus the equivalence class of X in the cokernel of j_k'' has finite order. This completes the proof. □

It remains to prove Lemma 3.6.3.

Proof of Lemma 3.6.3. Consider the coefficient of $[V_I] \in F_k'$ on the right-hand side of (3.20). The intersection number $[W_J] \cdot [V_{\overline{K}}]$ is non-zero if and only if the set K is obtained from J by removing a single element $j \in J$. Moreover, in this case we have $n \notin J$: otherwise we would conclude from the definition of E_k' that the set $J - \{n\}$ is long with respect to ℓ and then so would be the set $K = J - \{j\}$, contradicting the condition $[V_I] \in F_k$. Similarly, the intersection number $[W_{K'}] \cdot [V_{\overline{I}}]$ is non-zero if and only if I is obtained from K' by removing an element $i \in K'$. We note that $i \neq n$, since otherwise we would have $K = I$ contradicting the fact that F_k and F_k' are disjoint.

We see that a necessary condition for the coefficient of $[V_I]$ on the right-hand side of (3.20) to be non-zero is that the set the set I may be written as $I = J - \{i, j\} \cup \{n\}$ for some pair of indices $i, j \in J$, $i \neq j$. In this case the coefficient is given by

$$-\mu_i - \mu_j,$$

where

$$\mu_i = \frac{[W_J] \cdot [V_{\overline{I}_i}]}{[W_{I_i'}] \cdot [V_{\overline{I}_i}]} \frac{[W_{I_i'}] \cdot [V_{\overline{I}}]}{[V_I] \cdot [V_{\overline{I}}]}$$

and $I_i = I - \{n\} \cup \{i\}$; the number μ_j is defined analogously, but with the index i replaced by j. Since $\mu_i, \mu_j \in \{\pm 1\}$, every coefficient in the sum on the right-hand side of (3.20) is even. This establishes the first claim of Lemma 3.6.3.

We will show that

$$\mu_i = -\mu_j \text{ if } d \text{ is even} \tag{3.21}$$

and

$$\mu_i = \mu_j \text{ if } d \text{ is odd.} \tag{3.22}$$

This will complete the proof of Lemma 3.6.3. Indeed, from (3.21) we see that in the case where d is even, j_k'' vanishes identically. This establishes the second assertion of Lemma 3.6.3.

Equation (3.22) together with the definition of the coefficients μ_i and μ_j shows that if d is odd, then the homomorphism j''_k is not identically zero if and only if there exist basis elements $[W_J] \in E'_k$ and $[V_I] \in F'_k$ so that $n \notin J$ and $I = J - \{i, j\} \cup \{n\}$ for some pair of indices $i, j \in J$, $i \neq j$. Examining the definition of the subgroups E'_k and F'_k, one finds that this last condition is equivalent to condition (b) in the second part of Lemma 3.6.2.

Our proof of (3.21) and (3.22) relies on a symmetry argument. Consider the homeomorphism $\phi : W \to W$ which interchanges the ith and the jth factor of the product $W = (S^{d-1})^n$. Since the set J contains both indices i and j and neither of these two indices lies in I,

$$\phi(W_J) = W_J,\ \phi(V_I) = V_I \text{ and } \phi(V_{\bar{I}}) = V_{\bar{I}}. \tag{3.23}$$

Similarly,

$$\phi(V_{I'_i}) = V_{I'_j},\ \phi(V_{I'_j}) = V_{I'_i} \tag{3.24}$$

and

$$\phi(V_{\bar{I}_i}) = V_{\bar{I}_j},\ \phi(V_{\bar{I}_j}) = V_{\bar{I}_i}. \tag{3.25}$$

We also note that

$$\phi_*([W_J]) = [W_J]. \tag{3.26}$$

Assume that d is even. In this case ϕ reverses the orientation of W and thus for any two classes $x \in H_{(d-1)k}(W; \mathbf{Z})$ and $y \in H_{(d-1)(n-k)}(W; \mathbf{Z})$,

$$\phi_*(x) \cdot \phi_*(y) = -x \cdot y. \tag{3.27}$$

Moreover, using our choice of the orientations of the submanifolds V_I,

$$\phi_*([V_I]) = -[V_I]. \tag{3.28}$$

We compute

$$\mu_i = \frac{\phi_*([W_J]) \cdot \phi_*([V_{\bar{I}_i}])}{\phi_*([W_{I'_i}]) \cdot \phi_*([V_{\bar{I}_i}])} \frac{\phi_*([W_{I'_i}]) \cdot \phi_*([V_{\bar{I}}])}{\phi_*([V_I]) \cdot \phi_*([V_{\bar{I}}])} \tag{3.29}$$

$$= -\frac{[W_J]\cdot[V_{\bar{I}_j}]}{[W_{I'_j}]\cdot[V_{\bar{I}_j}]}\frac{[W_{I'_j}]\cdot[V_{\bar{I}}]}{[V_I]\cdot[V_{\bar{I}}]} = -\mu_j. \tag{3.30}$$

Here (3.29) follows from (3.27) and (3.30) from (3.23)-(3.26) and (3.28). Note that equations (3.23)-(3.25) imply that

$$\phi_*([V_{\bar{I}}]) = \pm[V_{\bar{I}}],\ \phi_*([V_{\bar{I}_i}]) = \pm[V_{\bar{I}_j}] \text{ and } \phi_*([V_{I'_i}]) = \pm[V_{I'_j}]. \tag{3.31}$$

Each of the classes $\phi_*([V_{\bar{I}}])$, $\phi_*([V_{\bar{I}_i]})$ and $\phi_*([V_{I'_i]})$ enters the expression on the right-hand side of (3.29) twice. Thus the signs carried over from the right-hand sides in (3.31) cancel.

Suppose now that d is odd. In this case ϕ preserves the orientation of W and thus for $x \in H_{(d-1)k}(W;\mathbf{Z})$ and $y \in H_{(d-1)(n-k)}(W;\mathbf{Z})$,

$$\phi_*(x)\cdot\phi_*(y) = x\cdot y. \tag{3.32}$$

Moreover,

$$\phi_*([V_I]) = [V_I]. \tag{3.33}$$

Arguing as in the above proof of (3.21), one obtains

$$\begin{aligned}\mu_i &= \frac{\phi_*([W_J])\cdot\phi_*([V_{\bar{I}_i}])}{\phi_*(([W_{I'_i}])\cdot\phi_*([V_{\bar{I}_i}])}\frac{\phi_*([W_{I'_i}])\cdot\phi_*([V_{\bar{I}}])}{\phi_*([V_I])\cdot\phi_*([V_{\bar{I}}])}\\ &= \frac{[W_J]\cdot[V_{\bar{I}_j}]}{[W_{I'_j}]\cdot[V_{\bar{I}_j}]}\frac{[W_{I'_j}]\cdot[V_{\bar{I}}]}{[V_I]\cdot[V_{\bar{I}}]} = \mu_j.\end{aligned}$$

This completes the proof. □

Chapter 4

Homology of planar telescopic Polygons

In this chapter we compute the homology groups of spaces of polygons with a telescopic edge in the planar case $d = 2$. We discuss an application motivated by the Topological Hypothesis studied in the theory of phase transitions.

4.1 The Homology Groups

The purpose of this section is to present a computation of the homology groups of spaces of planar polygonal linkages with a segment of variable length. Let us recall the construction of these spaces from Section 1.1. We fix length vectors ℓ^- and ℓ^+ of the form

$$\ell^- = (l_1, \ldots, l_{n-1}, l_n^-)$$

and

$$\ell^+ = (l_1, \ldots, l_{n-1}, l_n^+),$$

where $l_n^- < l_n^+$, and consider the closed interval $A \subset \mathbb{R}^n$ connecting ℓ^- and ℓ^+. The space $E_d(A)$ of polygons in Euclidean space $\mathbb{R}^d$ with $n-1$ edges of fixed lengths $l_1, \ldots, l_{n-1}$ and an edge whose length varies in the interval $[l_n^-, l_n^+]$ is defined as the union

$$E_d(A) = \cup_{\ell \in A} E_d(\ell) \subset (S^{d-1})^n.$$

Using the second part of Proposition 1.5.5, we can assume without loss of generality that the metric data A satisfies the condition $l_1 \leq l_2 \leq \cdots \leq l_{n-1}$.

Our study of the spaces $E_d(A)$ is motivated by the fact that they may be viewed as configuration spaces of linkages equipped with a telescopic segment. Telescopic legs are used quite commonly in mechanical linkages for example for shock absorption. As we show in Section 4.3, the spaces $E_d(A)$ are also of interest for certain topological questions studied in the thermodynamics literature.

Let us consider the space $E_d(A)$ in the case $d = 2$. There is a free $SO(2)$-action on $E_2(A)$ and the same argument as used in the proof of the first part of Proposition 1.5.2 shows:

Proposition 4.1.1. *If M_A denotes the quotient space $M_A = E_2(A)/SO(2)$, then there is a homeomorphism*

$$E_2(A) \simeq S^1 \times M_A.$$

It will be more convenient for us to study the spaces M_A rather than the products $E_2(A) \simeq S^1 \times M_A$.

The main result of this chapter is the computation of the homology groups of the spaces M_A.

Theorem 4.1.2. *Let A be the metric data of a polygon telescopic linkage with $n-1$ segments of fixed lengths $l_1 \leq l_2 \leq \cdots \leq l_{n-1}$ and one telescopic segment whose length varies in the interval $[l_n^-, l_n^+]$, $0 < l_n^- < l_n^+$. Assume that A is generic (see Section 1.5). Then for every $k = 0, \ldots, n-2$ the homology group $H_k(M_A; \mathbf{Z})$ is free abelian with rank*

$$\alpha_k(\ell^-) - \beta_k(\ell^+, \ell^-) + \alpha_{n-3-k}(\ell^+) - \beta_{n-3-k}(\ell^-, \ell^+).$$

We refer to Section 1.4 for the definition of the numbers α_k and β_k.

In the next two sections, applications of Theorem 4.1.2 are discussed.

4.2 Examples and Applications

In this section we evaluate Theorem 4.1.2 to study the topology of the spaces M_A.

We assume throughout this section that the metric data A is generic and satisfies the condition

$$l_1 \leq l_2 \leq \cdots \leq l_{n-1}.$$

Example 4.2.1. Consider the case where the interval of variation of the length of the telescopic segment is small. Namely, let $\ell = (l_1, \ldots, l_n)$ be a generic length vector and assume that

$$\ell^- = (l_1, \ldots, l_{n-1}, l_n - \varepsilon)$$

and

$$\ell^+ = (l_1, \ldots, l_{n-1}, l_n + \varepsilon),$$

where $0 < \varepsilon < \min(l_n, [\ell])$. Theorem 4.1.2 and the third part of Proposition 1.4.4 imply that for $0 \leq k \leq n-2$ the homology group $H_k(M_A; \mathbf{Z})$ is free abelian of rank

$$\text{rk } H_k(M_A; \mathbf{Z}) = a_k(\ell) + a_{n-3-k}(\ell).$$

On the other hand, it follows from Proposition 1.5.6 that in this case there is homeomorphism

$$M_A \simeq M_\ell \times [-\varepsilon, \varepsilon].$$

We see that for generic length vectors Theorem 4.1.2 recovers the computation of the integral homology groups of the spaces M_ℓ of planar polygons with fixed edge lengths obtained in [15].

Example 4.2.2. Let us assume the inequalities (a) $l_{n-1} > l_1 + \cdots + l_{n-2}$, (b) $0 < l_n^- < l_{n-1} - (l_1 + \cdots + l_{n-2})$ and (c) $l_n^+ > l_1 + \cdots + l_{n-1}$.

Recall from the proof of Proposition 1.5.5 in Section 1.6 that the space $E_2(A)$ may be identified as

$$E_2(A) = f_{\ell'}^{-1}([a, b]),$$

where ℓ' denotes the length vector $\ell' = (l_1, \ldots, l_{n-1})$, $f_{\ell'} : (S^1)^{n-1} \to \mathbb{R}$ is the corresponding robot arm distance map and $a = -(l_n^+)^2$, $b = -(l_n^-)^2$. The minimum of the function $f_{\ell'}$ is given by $-(l_1+\cdots+l_{n-1})^2$; moreover, assuming inequality (a) its maximum is $-(l_{n-1}-(l_1+\cdots+l_{n-2}))^2$. Using inequalities (b) and (c), it follows that in this case there are homeomorphisms $E_2(A)\simeq(S^1)^{n-1}$ and $M_A\simeq(S^1)^{n-2}$.

Inequality (b) implies that a subset $J\subset\{1,\dots,n\}$ is short with respect to ℓ^- if and only if J does not contain the index $n-1$. Thus

$$\alpha_k(\ell^-)=\binom{n-2}{k}\text{ for }k=0,\dots,n-2.$$

By inequality (c) every subset $J\subset\{1,\dots,n\}$ with $n\in J$ is long with respect to ℓ^+. Thus $\alpha_k(\ell^+)=0$ for all k. Similarly, the numbers $\beta_k(\ell^+,\ell^-)$ and $\beta_k(\ell^-,\ell^+)$ all vanish. We see that the result of Theorem 4.1.2 is consistent with the homeomorphism $M_A\simeq(S^1)^{n-2}$.

Example 4.2.3. We now study connectedness of the spaces M_A. By Theorem 4.1.2, the zero Betti number of M_A is

$$\text{rk }H_0(M_A;\mathbf{Z})=\alpha_0(\ell^-)-\beta_0(\ell^+,\ell^-)+\alpha_{n-3}(\ell^+)-\beta_{n-3}(\ell^-,\ell^+).$$

From the definition of the numbers α_k and β_k we see that the difference $\alpha_0(\ell^-)-\beta_0(\ell^+,\ell^-)$ can assume the values zero or one. Moreover, if $\alpha_0(\ell^-)-\beta_0(\ell^+,\ell^-)=0$, then either the set $\{n\}$ is long with respect to ℓ^- or the set $\{n-1\}$ is long with respect to ℓ^+. Thus using the second part of Proposition 1.5.1, $\alpha_0(\ell^-)-\beta_0(\ell^+,\ell^-)=1$ if the space M_A is non-empty.

The difference $\alpha_{n-3}(\ell^+)-\beta_{n-3}(\ell^-,\ell^+)$ is the number of two-element subsets $J\subset\{1,\dots,n-1\}$ which are long with respect to ℓ^+ and satisfy one of the following two conditions: either (a) $n-1\notin J$ or (b) $n-1\in J$ and the set $J\cup\{n\}-\{n-1\}$ is long with respect to ℓ^-. We claim that there is at most one two-element subset $J\subset\{1,\dots n-1\}$ which is long with respect to ℓ^+ and satisfies (a) or (b). Thus the number $\alpha_{n-3}(\ell^+)-\beta_{n-3}(\ell^-,\ell^+)$ can assume the values zero or one.

Due to the assumption $l_1\le l_2\le\cdots\le l_{n-1}$, either l_{n-1} or l_n^+ is a maximal entry of ℓ^+. Since a length vector admits at most one long two-element set not containing the index of a maximal entry, there can be at most one two-element subset

$J \subset \{1, \ldots n-1\}$ which is long with respect to ℓ^+ and satisfies condition (a), namely $J = \{n-3, n-2\}$. If the set $\{n-3, n-2\}$ is long with respect to ℓ^+, then it is also long with respect to ℓ^-. It follows that the set $\{n-1, n\}$ is short with respect to ℓ^-, but then so is the set $\{n-2, n\}$. We conclude that in this case there are no two-element subsets $J \subset \{1, \ldots, n-1\}$ that are long with respect to ℓ^+ and satisfy condition (b).

Assume now that there is a two-element subset $J \subset \{1, \ldots, n-1\}$ which is long with respect to ℓ^+ and satisfies condition (b). Then the set $\{n-2, n\}$ is long with respect to ℓ^- and therefore also long with respect to ℓ^+. Thus in this case no two-element subset $J \subset \{1, \ldots, n-1\}$ which is long with respect to ℓ^+ satisfies condition (a).

It follows that the space M_A has at most two connected components and is disconnected if and only if either the condition

$$\{n-3, n-2\} \text{ is long w.r.t. } \ell^+ \tag{4.1}$$

or the two conditions

$$\{n-2, n-1\} \text{ is long w.r.t. } \ell^+ \text{ and } \{n-2, n\} \text{ is long w.r.t. } \ell^- \tag{4.2}$$

are satisfied. One concludes:

Proposition 4.2.4. *If the space M_A is disconnected, then there exist three indices $1 \leq i < j < k \leq n$ so that for every length vector $\ell \in A$ the two-elements sets $\{i, j\}$, $\{i, k\}$ and $\{j, k\}$ are long with respect to ℓ.*

Proof. If condition (4.1) is satisfied, then indices i, j, k with the properties as indicated in the claim of the Proposition are given by $i = n-3$, $j = n-2$ and $k = n-1$. In the case when the conditions of (4.2) are met, we may set $i = n-2$, $j = n-1$ and $k = n$. $\square$

Proposition 4.2.4 means that if the space M_A is disconnected, then for each $\ell \in A$ there is a massive triangle (see Proposition 1.6.6). Using Proposition 1.6.6 with $d = 2$, it follows that in this case for every $\ell \in A$ the space $E_2(\ell)$, and thus also the space M_ℓ is disconnected. We have shown:

Corollary 4.2.5. *If the space M_A is disconnected, then so are all the spaces M_ℓ where $\ell \in A$.*

Example 4.2.6. The following example shows that the space M_A may be connected although both ends $M_{\ell^\pm}$ are disconnected. Let $n = 4$, $l_1 = 4$, $l_2 = 8$, $l_3 = 10$, $l_4^- = 1$ and $l_4^+ = 12$. In this case the spaces M_{ℓ^-} and M_{ℓ^+} are disconnected because a massive triangle exists both for ℓ^- as well as for ℓ^+. However, the space M_A is connected as neither (4.1) nor (4.2) is satisfied.

4.3 Telescopic Linkages and the Topological Hypothesis

In this section we discuss an application of the study of the topology of telescopic polygons motivated by the so-called *Topological Hypothesis* in the theory of phase transitions.

The Topological Hypothesis consists of the statement that phase transitions in thermodynamical systems are caused by changes in the topology of certain submanifolds in the configuration space (see e.g. [5], [35] and references therein).

We consider a system with a Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + V(q_1, \ldots, q_N).$$

It is known that at any given value of the inverse temperature β, the effective support of the canonical measure is close to an equipotential surface

$$\sigma_v = \{q \in \Gamma_N : V(q) = vN\}.$$

Here Γ_N is the configuration space and V the potential of the system.

The Topological Hypothesis states that the reason for the singular behaviour of thermodynamic observables in phase transitions of the system are changes of the topology of the level sets Σ_v or of the sublevels

$$\mathcal{M}_v = \{q \in \Gamma_N : V(q) \leq vN\}. \tag{4.3}$$

In [16] a version of the Topological Hypothesis was proved for a class of short-range models. Other known results concern the asymptotic growth rate

$$\sigma(v) = \lim_{N\to\infty} \frac{1}{N} \ln |\chi(\mathcal{M}_v)| \tag{4.4}$$

of the Euler characteristic of the sublevels. In many cases non-smoothness of the function $\sigma(v)$ detects the presence of a phase transition (see e.g. the exposition in [28]). In the literature there are also results of negative character concerning the Topological Hypothesis ([29],[38]).

We now show that the spaces M_A of planar polygons with a telescopic segment studied in the previous section can be identified with the sublevels $\mathcal{M}_v$ of a known thermodynamical model.

We consider the anti-ferromagnetic mean-field XY-model ([6], [32]), which is characterized by the potential

$$V = \frac{1}{2N} \sum_{j,k=1}^{N} cos(\theta_j - \theta_k) - h \sum_{j=1}^{N} \theta_j, \tag{4.5}$$

where $\theta_i \in [0, 2\pi]$ are angular parameters and $h > 0$ is also referred to as an external magnetic field. The configuration space Γ_N of the model is an N-torus T^N and we write points $q \in \Gamma_N$ as $q = \{e^{i\theta_1}, \ldots, e^{i\theta_N}\}$.

Denoting

$$\mathbf{m} = \frac{1}{N} \sum_{j=1}^{N} exp\ i\theta_j$$

and

$$\mathbf{m}_0 = -ih,$$

one obtains

$$|\mathbf{m} + \mathbf{m}_0|^2 = \frac{2}{N} V + h^2.$$

Thus the sublevel $\mathcal{M}_v$ defined by (4.3) can be identified as

$$\mathcal{M}_v = \{q : |\mathbf{m} + \mathbf{m}_0|^2 \leq 2v + h^2\}. \tag{4.6}$$

The interval (a_h, b_h) of variation of the parameter v is given by

$$b_h = h + 1/2$$

and

$$a_h = \begin{cases} -\frac{h^2}{2} & \text{if } h \in (0, 1], \\ -h + \frac{1}{2} & \text{if } h \in [1, \infty). \end{cases}$$

Comparing (4.6) with the definition of the space M_A of planar telescopic polygons, one obtains:

Proposition 4.3.1. *The space $\mathcal{M}_v$ defined by (4.3) is homeomorphic to the space M_A of planar polygons with $N + 1$ segments of fixed lengths $l_1 = \cdots = l_N = 1/N$, $l_{N+1} = h$, and a telescopic segment whose length varies in the interval* $[0, (2v+h^2)^{1/2}]$.

Our goal is to study the total Betti numbers

$$b(\mathcal{M}_v) = \sum_{k=0}^{N+2} b_k(\mathcal{M}_v), \tag{4.7}$$

where $b_k(\mathcal{M}_v) = \mathrm{rk}\ H_k(\mathcal{M}_v; \mathbf{Z})$, and their exponential growth rate

$$\tau(v) = \lim_{N \to \infty} \frac{1}{N} \ln b(\mathcal{M}_v). \tag{4.8}$$

We want to apply the identification given in Proposition 4.3.1 of the sublevels (4.3) with the configuration spaces of a planar telescopic linkage. The formulae of Proposition 4.1.2 cannot be used to compute the homology groups in the case of the telescopic linkage defined in Proposition 4.3.1: Proposition 4.1.2 only applies in the case of a telescopic segment whose length varies between two positive numbers. However, the method of proof of Proposition 4.1.2 can be adapted to compute the Betti numbers of the sublevels $\mathcal{M}_v$. We will show:

Proposition 4.3.2. *Let $h > 0$ and $v \in (a_h, b_h)$. The exponential growth rate of the total Betti number (4.7) is*

$$\tau(v) = \begin{cases} -p_v \ln p_v - (1 - p_v) \ln(1 - p_v) & \text{if } v \leq 0 \\ \ln 2 & \text{if } v \geq 0. \end{cases} \tag{4.9}$$

Here $p_v = \frac{1}{2}((2v + h^2)^{1/2} - h + 1)$.

Proposition 4.3.2 shows that the function $\tau(v)$ and its first derivative are continuous at $v = 0$, however the second derivative is discontinuous at that point. In light of the Topological Hypothesis, it is an interesting question how this result, obtained purely by topological methods, is reflected by the physics of the model.

4.4 Proof of Theorem 4.1.2

The space M_A is a subset of

$$W = \{(u_1, \ldots, u_{n-1}) \in S^1 \times \cdots \times S^1\}/SO(2) \simeq T^{n-2}.$$

We can view W as the space of all *shapes* of a planar robot arm, i.e. configurations viewed up to rotations (compare with Section 1.2).

Similarly to the proof of Theorem 3.1.1, consider the function $f_\ell : W \to \mathbb{R}$ given by

$$f_\ell(u_1, \ldots, u_{n-1}) = -|\sum_{j=1}^{n-1} l_j u_j|^2.$$

The space M_A is the preimage $f_\ell^{-1}[a, b]$, where

$$a = -(l_n^+)^2 \text{ and } b = -(l_n^-)^2.$$

Denote

$$W^a = f_\ell^{-1}(-\infty, a],\ W^b = f_\ell^{-1}(-\infty, b] \text{ and } W^{[a,b]} = f_\ell^{-1}[a, b].$$

Our goal is to compute the homology groups of $W^{[a,b]} = M_A$. The computation will use an explicit description of the homology groups of the manifolds W^a and W^b which follows from the results of [15] (see also Corollary 1.19 in [7]).

For every subset $J \subset \{1, \ldots, n\}$, denote by $W_J \subset W$ the submanifold consisting of those tuples $(u_1, \ldots, u_n) \in W$, so that $u_i = u_j$ for all $i, j \in J$. Each manifold W_J is diffeomorphic to a torus of dimension $n - 1 - |J|$. Moreover, W_J is contained in W^a if and only if the set J is long with respect to ℓ^+. Similarly, $W_J \subset W^b$ if and only if J is long with respect to ℓ^-.

We choose orientations of W and of the submanifolds W_J as follows. For $j = 1, \dots, n-1$, denote by e_j the vector field on the torus T^{n-1} which is tangent to the jth circle and rotates in the positive direction. If $e'_1, \dots, e'_{n-1}$ denote the images of $e_1, \dots, e_{n-1}$ under the projection $T^{n-1} \to T^{n-1}/SO(2) = W$, then the vector fields $e'_1, \dots, e'_{n-1}$ generate at every point the tangent space to W and satisfy $e'_1 + \cdots + e'_{n-1} = 0$. We orient W by defining the basis $e'_2, \dots, e'_{n-1}$ to be positive.

Given a subset $J \subset \{1, \dots, n\}$, let $\overline{J} = \{k_1, \dots, k_r\}$, $k_1 < \cdots < k_r$ be the complement of J, where $r = n - 1 - |J|$. The vector fields $e'_{k_1}, \dots, e'_{k_r}$ form a basis of the tangent space of W_J at every point. We orient W_J so that this basis is positive.

Proposition 4.4.1 ([15]). *Let $0 \le k \le n-2$. The homology classes of all the submanifolds W_J, where $J \subset \{1, \dots, n-1\}$ is a subset of cardinality $|J| = n-1-k$ which is long with respect to ℓ^+, form a free basis of $H_k(W^a; \mathbf{Z})$. Similarly, the classes of the submanifolds W_J so that $J \subset \{1, \dots, n-1\}$, $|J| = n-1-k$ and J is long with respect to ℓ^-, form a free basis of $H_k(W^b; \mathbf{Z})$.*

It follows from Proposition 4.4.1 that

$$\operatorname{rk} H_k(W^a; \mathbf{Z}) = \alpha_k(\ell^+) \text{ and } \operatorname{rk} H_k(W^b; \mathbf{Z}) = \alpha_k(\ell^-),$$

where $\alpha_k(\ell^+)$ is the number of all subsets $J \subset \{1, \dots, n-1\}$ of cardinality $|J| = n-1-k$ which are long with respect to ℓ^+ (see Section 1.4).

Consider the homological exact sequence of the pair $(W^b, W^{[a,b]})$:

$$\cdots \to H_{k+1}(W^b, W^{[a,b]}; \mathbf{Z}) \to H_k(W^{[a,b]}; \mathbf{Z})$$
$$\to H_k(W^b; \mathbf{Z}) \xrightarrow{j_k} H_k(W^b, W^{[a,b]}; \mathbf{Z}) \to \dots$$

By Proposition 4.4.1, the integral homology groups of the spaces W^a and of W^b are free abelian. Thus using excision, Poincaré-Lefschetz duality and the universal coefficient theorem,

$$H_k(W^b, W^{[a,b]}; \mathbf{Z}) \simeq H^{n-2-k}(W^a; \mathbf{Z}) \simeq (H_{n-2-k}(W^a; \mathbf{Z}))^*, \tag{4.10}$$

where

$$(H_{n-2-k}(W^a;\mathbf{Z}))^* = \mathrm{Hom}(H_{n-2-k}((W^a;\mathbf{Z})),\mathbf{Z}).$$

Consider the intersection form

$$H_k(W^b;\mathbf{Z}) \otimes H_{n-2-k}(W^a;\mathbf{Z}) \to \mathbf{Z}. \tag{4.11}$$

Under the identifications of (4.10), the homomorphism

$$H_k(W^b;\mathbf{Z}) \to (H_{n-2-k}(W^a;\mathbf{Z}))^*$$

associated to the bilinear form (4.11) coincides with

$$j_k : H_k(W^b;\mathbf{Z}) \to H_k(W^b, W^{[a,b]};\mathbf{Z}).$$

There is a short exact sequence

$$0 \to \mathrm{coker}(j_{k+1}) \to H_k(W^{[a,b]};\mathbf{Z}) \to \ker(j_k) \to 0. \tag{4.12}$$

It r_k denotes the rank of the intersection form (4.11), then

$$r_k + \mathrm{rk}\ \ker(j_k) = \mathrm{rk}\ H_k(W^b;\mathbf{Z})$$

and

$$r_k + \mathrm{rk}\ \mathrm{coker}(j_k) = \mathrm{rk}\ H_k(W^b, W^{[a,b]};\mathbf{Z}) = \mathrm{rk}\ H_{n-2-k}(W^a;\mathbf{Z}).$$

Together with (4.12), it follows that

$$\mathrm{rk}\ H_k(M_A;\mathbf{Z}) = \mathrm{rk}\ H_k(W^{[a,b]};\mathbf{Z})$$

$$= \mathrm{rk}\ H_k(W^b;\mathbf{Z}) + \mathrm{rk}\ H_{n-3-k}(W^a;\mathbf{Z}) - r_k - r_{k+1}$$

$$= \alpha_k(\ell^-) + \alpha_{n-3-k}(\ell^+) - r_k - r_{k+1}.$$

Evidently, $\ker(j_k)$ is free abelian. Thus the short exact sequence (4.12) splits and the homology group $H_k(M_A;\mathbf{Z})$ is torsion-free if and only if so is the cokernel of the homomorphism j_{k+1}.

In order to compute the numbers r_k, we describe the intersection form (4.11). For this purpose, write $H_k(W^b;\mathbf{Z})$ as a direct sum

$$H_k(W^b;\mathbf{Z}) = A_k^b \oplus B_k^b \oplus C_k^b \tag{4.13}$$

of free abelian groups as follows:

- The subgroup A_k^b is generated by the homology classes of all the submanifolds W_J where $J \subset \{1, \ldots, n-1\}$, $|J| = n-1-k$, J is long with respect to ℓ^- and the set $\widehat{J}$ obtained from J by removing the maximal index lying in J and adding the index n is long with respect to ℓ^+.

- We define B_k^b as the subgroup generated by the classes of the submanifolds W_J so that $J \subset \{1, \ldots, n-1\}$, $|J| = n-1-k$, J is long with respect to ℓ^-, $n-1 \in J$ and $\widehat{J}$ is short with respect to ℓ^+ (note that in this case $\widehat{J} = J - \{n-1\} \cup \{n\}$).

- The subgroup C_k^b is generated by the homology classes of all the submanifolds W_J where $J \subset \{1, \ldots, n-2\}$, $|J| = n-1-k$, J is long with respect to ℓ^- and $\widehat{J}$ is short with respect to ℓ^+.

We write the homology group $H_k(W^a; \mathbf{Z})$ as a direct sum

$$H_k(W^a; \mathbf{Z}) = A_k^a \oplus B_k^a \oplus C_k^a, \tag{4.14}$$

of free abelian groups defined as above, but with the roles of ℓ^- and ℓ^+ interchanged. Thus

- A_k^a is generated by the classes $[W_J]$ so that $J \subset \{1, \ldots, n-1\}$, $|J| = n-1-k$, J is long with respect to ℓ^+ and $\widehat{J}$ is long with respect to ℓ^-.

- B_k^a is subgroup generated by the classes $[W_J]$ so that $J \subset \{1, \ldots, n-1\}$, $|J| = n-1-k$, J is long with respect to ℓ^+, $n-1 \in J$ and $\widehat{J}$ is short with respect to ℓ^-.

- C_k^a is generated by the classes $[W_J]$ so that $J \subset \{1, \ldots, n-2\}$, $|J| = n-1-k$, J is long with respect to ℓ^+ and $\widehat{J}$ is short with respect to ℓ^+.

Counting the numbers of the basis elements of the groups B_k^b and B_k^a and comparing with the combinatorial quantities introduced in Section 1.4, one obtains

$$\text{rk } B_k^b = \beta_k(\ell^+, \ell^-) \text{ and rk } B_k^a = \beta_k(\ell^-, \ell^+).$$

Thus to establish the claim of Theorem 4.1.2, it suffices to show that the cokernel of the homomorphism j_k is torsion-free and the rank of its image coincides with the rank of B_k^b.

We now evaluate the intersection form (4.11) on the direct summands of the splittings (4.13) and (4.14). If $J, K \subset \{1, \ldots, n-1\}$ are subsets with $|J| + |K| = n$, then the submanifolds W_J and W_K of W have complimentary dimensions and the intersection number of the homology classes $[W_J]$ and $[W_K]$ is

$$[W_J] \cdot [W_K] = \begin{cases} \pm 1 & \text{if } |J \cap K| = 1, \\ 0 & \text{if } |J \cap K| > 1. \end{cases}$$

Let $[W_J] \in A_k^b$ and $[W_K] \in H_{n-2-k}(W^a; \mathbf{Z})$. Suppose that the intersection number $[W_J] \cdot [W_K]$ is non-zero. Then the set K is obtained from the complement of J in $\{1, \ldots, n-1\}$ by adding an element $j \in J$. Since the set K is long with respect to ℓ^+, its complement

$$\overline{K} = \{1, \ldots, n\} - K$$

is short with respect to ℓ^+. As $\overline{K} = J - \{j\} \cup \{n\}$, it follows that the set $\widehat{J}$ obtained from J by removing its largest element is also short with respect to ℓ^+. This is a contradiction to the definition of the subgroup A_k^b. We see that the intersection number $[W_J] \cdot [W_K]$ of any two classes $[W_J] \in A_k^b$ and $[W_K] \in H_{n-2-k}(W^a; \mathbf{Z})$ vanishes. An analogous argument shows that $[W_J] \cdot [W_K] = 0$ for all $[W_J] \in H_k(W^b; \mathbf{Z})$ and $[W_K] \in A_{n-2-k}^a$.

Consider now the case $[W_J] \in B_k^b$ and $[W_K] \in B_{n-2-k}^a$. As both sets J and K contain the index $n-1$, the intersection number $[W_J] \cdot [W_K]$ is non-zero if and only if the set J is obtained from the complement $\overline{K}$ by removing the index n and adding the index $n-1$. Examining the definitions of the subgroups B_k^b and B_{n-2-k}^a, one concludes that for every class $[W_J] \in B_k^b$, there is a unique basis element $[W_K] \in B_{n-2-k}^a$ with $[W_J] \cdot [W_K] \neq 0$ (given by $K = \overline{J} - \{n\} \cup \{n-1\}$) and the intersection number with that basis element is $[W_J] \cdot [W_K] = \pm 1$. It follows that the restriction of the intersection form (4.11) to $B_k^b \otimes B_{n-2-k}^a$ is nondegenerate.

The intersection number $[W_J] \cdot [W_K]$ vanishes if $[W_J] \in C_k^b$ and $[W_K] \in C_{n-2-k}^a$.

Indeed, since in this case neither the set J nor the set K contains the index $n-1$, the two sets have at least two elements in common.

We define for each basis element $[W_J] \in C_k^b$ an element $Y_J \in H_k(W^b; \mathbf{Z})$ by

$$Y_J = [W_J] - \sum_I \frac{[W_J] \cdot [W_{I'}]}{[W_I] \cdot [W_{I'}]} [W_I], \tag{4.15}$$

where the sum is over all the basis elements of the subgroup B_k^b and I' denotes the set

$$I' = \overline{I} - \{n\} \cup \{n-1\}.$$

Here $\overline{I}$ is the complement of I in $\{1, \ldots, n\}$.

It follows from (4.15) that the intersection number $Y_J \cdot [W_K]$ is zero for all $[W_K] \in A^a_{n-2-k} \oplus B^a_{n-2-k}$. Let us show that the intersection number also vanishes in the case $[W_K] \in C^a_{n-2-k}$.

The non-zero summands on the right-hand side of (4.15) correspond to sets I of the form

$$I = J - \{j\} \cup \{n-1\}, \tag{4.16}$$

where $j \in J$. We denote the right-hand side of (4.16) by $I_j = J - \{j\} \cup \{n-1\}$.

If $J \subset \{1, \ldots, n-2\}$ is long with respect to ℓ^- and $\widehat{J}$ is short with respect to ℓ^+, then for every $j \in J$ the set I_j is long with respect to ℓ^- and the set $\widehat{I_j}$ is short with respect to ℓ^+. Thus

$$Y_J = [W_J] - \sum_{j \in J} \frac{[W_J] \cdot [W_{I'_j}]}{[W_{I_j}] \cdot [W_{I'_j}]} [W_{I_j}]. \tag{4.17}$$

For $[W_K] \in C^a_{n-2-k}$, the intersection number $Y_J \cdot [W_K]$ can be expressed as

$$\begin{aligned} Y_J \cdot [W_K] &= [W_J] \cdot [W_K] - \sum_{j \in J} \frac{[W_J] \cdot [W_{I'_j}]}{[W_{I_j}] \cdot [W_{I'_j}]} ([W_{I_j}] \cdot [W_K]) \\ &= \sum_{j \in J} \frac{[W_J] \cdot [W_{I'_j}]}{[W_{I_j}] \cdot [W_{I'_j}]} ([W_{I_j}] \cdot [W_K]). \end{aligned} \tag{4.18}$$

If $|K \cap I_j| = 1$ for some $j \in J$, then $|J \cap K| = 2$. Thus in the case $|J \cap K| > 2$ every term in the sum on the right-hand side of (4.18) is zero and hence $Y_J \cdot [W_K] = 0$.

Assume now that $|J \cap K| = 2$. Let $J \cap K = \{i, j\}$. Then one obtains

$$Y_J \cdot [W_K] = -\nu_i - \nu_j,$$

where

$$\nu_i = \frac{[W_J] \cdot [W_{I'_i}]}{[W_{I_i}] \cdot [W_{I'_i}]}([W_{I_i}] \cdot [W_K])$$

and where ν_j is defined analogously, but with the index i replaced by j. To prove that the intersection number $Y_J \cdot [W_K]$ vanishes, we must demonstrate that $\nu_i = -\nu_j$. This will be shown by a similar symmetry argument as was used in the proof of Theorems 3.1.3 and 3.1.5 in the preceding chapter.

Consider the homeomorphism $T^{n-1} \to T^{n-1}$ which interchanges the ith and the jth coordinate and the induced homeomorphism $\phi : W \to W$. The fact that the sets J and K contain the indices i and j implies that

$$\phi(W_J) = W_J \text{ and } \phi(W_K) = W_K. \tag{4.19}$$

Moreover,

$$\phi(W_{I_i}) = W_{I_j},\ \phi(W_{I_j}) = W_{I_i} \tag{4.20}$$

and

$$\phi(W_{I'_i}) = W_{I'_j},\ \phi(W_{I'_j}) = W_{I'_i}. \tag{4.21}$$

Since ϕ reverses the orientation of W, for any two homology classes $x \in H_k(W; \mathbf{Z})$, $y \in H_{n-2-k}(W; \mathbf{Z})$,

$$\phi_*(x) \cdot \phi_*(y) = -x \cdot y. \tag{4.22}$$

As ϕ preserves the orientations of the submanifolds W_J and W_K,

$$\phi_*([W_J]) = [W_J] \text{ and } \phi_*([W_K]) = [W_K]. \tag{4.23}$$

We compute

$$\nu_i = -\frac{\phi_*([W_J]) \cdot \phi_*([W_{I'_i}])}{\phi_*([W_{I_i}]) \cdot \phi_*([W_{I'_i}])}(\phi_*([W_{I_i}]) \cdot \phi_*([W_K])) \tag{4.24}$$

$$= -\frac{[W_J] \cdot [W_{I'_j}]}{[W_{I_j}] \cdot [W_{I'_j}]}([W_{I_j}] \cdot [W_K]) = -\nu_j. \tag{4.25}$$

Here (4.24) follows from (4.22) and (4.25) is obtained with the help of (4.19), (4.21) and (4.23). We note that by (4.19) and (4.21),

$$\phi_*([W_{I_i}]) = \pm[W_{I_j}] \text{ and } \phi_*([W_{I'_i}]) = \pm[W_{I'_j}]. \tag{4.26}$$

One observes that since each of the classes $\phi_*([W_{I_i}])$ and $\phi_*([W_{I'_i}])$ appears twice on the right-hand side of (4.24), the signs from the right-hand sides of (4.26) cancel.

We can now complete the proof of Theorem 4.1.2. If $\widetilde{C}_k^b \subset H_k(W^b; \mathbf{Z})$ denotes the subgroup generated by all the homology classes Y_J corresponding to basis elements $[W_J] \in C_k^b$, then there is a direct sum decomposition

$$H_k(W^b; \mathbf{Z}) = A_k^b \oplus B_k^b \oplus \widetilde{C}_k^b.$$

We have shown that the homomorphism

$$j_k : H_k(W^b; \mathbf{Z}) \to H_k(W^b, W^{[a,b]}) \simeq (H_{n-2-k}(W^a; \mathbf{Z}))^*$$

vanishes on $A_k^b \oplus \widetilde{C}_k^b$ and the restriction of j_k to B_k^b is a monomorphism onto a direct summand. It follows that the cokernel of the homomorphism j_k is torsion-free and the rank of the image of j_k coincides with the rank of B_k^b. This completes the proof.

4.5 Proof of Proposition 4.3.2

The claim of Proposition 4.3.2 will be concluded from the following result:

Proposition 4.5.1. *Let A be the metric data of the polygonal telescopic linkage defined in Proposition 4.3.1. Denote*

$$c_k = \begin{cases} \binom{N}{k} & \text{if} \quad 0 \le k \le p_v N, \\ 0 & \text{if} \quad p_v N < k \le N. \end{cases} \tag{4.27}$$

Moreover, define

$$d_k = \begin{cases} \binom{N}{k-1} & \text{if} \quad 1 \le k < (1-h-p_v)N + 1, \\ 0 & \text{if} \quad (1-h-p_v)N + 1 \le k \le N-1. \end{cases} \tag{4.28}$$

and $d_0 = d_{-1} = 0$.

Let $N > h$. *Then for* $0 \leq k \leq N$, *the homology group* $H_k(M_A; \mathbf{Z})$ *is free abelian with rank*

$$rk\ H_k(M_A; \mathbf{Z}) = c_k + d_{N-1-k}.$$

Proof of Proposition 4.3.2. Denote

$$S_k^N = \sum_{0 \leq j \leq k} \binom{N}{j} \text{ and } R_k^N = \sum_{0 \leq j < k} \binom{N}{j}.$$

Combining Propositions 4.3.1 and 4.5.1, one obtains the following formula for the total Betti number of the sublevels $\mathcal{M}_v$ for $N > h$:

$$b(\mathcal{M}_v) = S_{p_v N}^N + R_{(1-h-p_v)N}^N.$$

Assume first that $v \leq 0$. In this case $1 - p_v - h \leq p_v \leq 1/2$ and hence

$$\binom{N}{\lfloor p_v N \rfloor} < b(\mathcal{M}_v) < 2S_{p_v N}^N \leq n \binom{N}{\lfloor p_v N \rfloor}. \tag{4.29}$$

To further analyze the total Betti number, the following asymptotic formula for the binomial coefficients will be used (see [4], page 4):

$$\binom{n}{m} \sim (2\pi)^{-1} \left(\frac{n}{m}\right)^m \left(\frac{n}{n-m}\right)^{n-m} \left(\frac{m(n-m)}{n}\right)^{-1/2}. \tag{4.30}$$

This last asymptotic formula holds if $n, m \to \infty$ and $n - m \to \infty$. The notation $f(n) \sim g(n)$ means that $f(n)/g(n) \to 1$.

Using (4.30), one obtains:

$$\lim_{N\to\infty} \frac{1}{N} \ln \binom{N}{\lfloor p_v N \rfloor} = -p_v \ln p_v - (1 - p_v) \ln(1 - p_v). \tag{4.31}$$

The first part of (4.9) follows by combining (4.29) and (4.31).

Consider now the case $v \geq 0$. Here $p_v \geq 1/2$ and thus $S_{p_v N}^N \geq 2^{N-1}$. We obtain the estimate

$$2^{N-1} \leq b(\mathcal{M}_v) \leq 2^{N+1}. \tag{4.32}$$

The second part of (4.9) follows from (4.32). $\square$

Proof of Proposition 4.5.1. Denote by ℓ the length vector with entries

$$l_1 = \cdots = l_N = \frac{1}{N},\ l_{N+1} = h \ \text{ and } \ l_{N+2} = (2v + h^2)^{1/2}. \tag{4.33}$$

Due to the assumption $N > h$, we have $l_j \leq l_{N+1}$ for $j = 1, \ldots, N+1$.

Consider

$$W = \{(u_1 \ldots u_{N+1}) \in (S^1)^{N+1}\}/SO(2) \simeq T^N$$

and the function $f_\ell : W \to \mathbb{R}$ defined in the previous section.

Let A be the metric data of the telescopic linkage defined in Proposition 4.3.1. Using similar notation as in the proof of Proposition 4.1.2, define

$$W^a = f_\ell^{-1}(-\infty, a],\ a = (2v + h^2)^{1/2}.$$

Consider the homology exact sequence of the pair (W, W^a):

$$\cdots \to H_{k+1}(W, W^a; \mathbf{Z}) \to H_k(W^a; \mathbf{Z}) \xrightarrow{i_k} H_k(W; \mathbf{Z}) \to H_k(W, W^a; \mathbf{Z}) \to \ldots$$

Using excision and Poincaré-Lefschetz duality,

$$\begin{aligned} H_{k+1}(W, W^a; \mathbf{Z}) &\simeq H_{k+1}(M_A, \partial M_A; \mathbf{Z}) \\ &\simeq H^{N-1-k}(M_A; \mathbf{Z}) \simeq H^{N-1-k}(\mathcal{M}_v; \mathbf{Z}). \end{aligned}$$

Thus there is a short exact sequence

$$0 \to \operatorname{coker}(i_{k+1}) \to H^{N-1-k}(\mathcal{M}_v; \mathbf{Z}) \to \ker(i_k) \to 0. \tag{4.34}$$

By Proposition 4.4.1, there is a free basis of $H_k(W^a; \mathbf{Z})$ consisting of the homology classes $[W_J]$, where $J \subset \{1, \ldots, N+1\}$ is a subset of cardinality $|J| = N + 1 - k$ which is long with respect to ℓ. Consider the direct sum decomposition

$$H_k(W^a; \mathbf{Z}) = D_k \oplus E_k, \tag{4.35}$$

where D_k is the subgroup generated by the homology classes $[W_J]$ so that $J \subset \{1, \ldots, N+1\}$, $|J| = N + 1 - k$, J is long with respect to ℓ and contains the index $N+1$. The subgroup E_k is generated by the classes $[W_J]$ so that $J \subset \{1, \ldots, N+1\}$,

$|J| = N+1-k$, J is long with respect to ℓ and $N+1 \notin J$.

The homology group $H_k(W;\mathbf{Z})$ has a free basis consisting of all the classes $[W_J]$ where $J \subset \{1,\dots,N+1\}$, $|J| = N+1-k$ and $N+1 \in J$ (we refer to the previous section for the definition of the submanifolds W_J). We write $H_k(W;\mathbf{Z})$ as a direct sum

$$H_k(W;\mathbf{Z}) = D_k \oplus F_k, \tag{4.36}$$

of free abelian groups, where D_k is defined as above and F_k is the subgroup generated by the classes $[W_J]$ so that $J \subset \{1,\dots,N+1\}$, $|J| = N+1-k$, $N+1 \in J$ and so that J is either short or median with respect to ℓ.

The homomorphism

$$i_k : H_k(W^a;\mathbf{Z}) \to H_k(W;\mathbf{Z})$$

induced by inclusion restricts to the identity on D_k. Let us show that $i_k(E_k) \subset D_k$.

Consider an element $[W_J]$ of the specified basis of E_k. We compute

$$i_k([W_J]) = \sum_{[W_I]\in D_k} \frac{[W_J]\cdot[W_{I'}]}{[W_I]\cdot[W_{I'}]}[W_I] + \sum_{[W_K]\in F_k} \frac{[W_J]\cdot[W_{K'}]}{[W_K]\cdot[W_{K'}]}[W_K]. \tag{4.37}$$

Here I' denotes the set $\{1,\dots,N+1\} - I \cup \{N+1\}$.

Suppose that for some class $[W_K] \in F_k$ coefficient of $[W_K]$ in the second sum on the right-hand side of (4.37) is non-zero. Then the sets J and K' must have a unique common element $j \in \{1,\dots,N+1\}$. Thus the set K is obtained from J by removing the index j and adding the index $N+1$. Since $l_j \le l_{N+1}$ and J is long with respect to ℓ, it follows that K must also be long with respect to ℓ. This contradicts the definition of the subgroup F_k. We conclude that the second sum on the right-hand side of (4.37) vanishes identically for every class $[W_J] \in E_k$. Thus $i_k(E_k) \subset D_k$.

It follows from the above discussion that the ranks of the kernel and of the cokernel of i_k coincide with the ranks of E_k and of F_k respectively. Moreover, the cokernel of i_k is torsion-free. Using the short exact sequence (4.34), we conclude that $H^{N-1-k}(\mathcal{M}_v; \mathbf{Z})$ is the free abelian group of rank

$$\text{rk } H^{N-1-k}(\mathcal{M}_v; \mathbf{Z}) = \text{rk } E_k + \text{rk } F_{k+1}. \tag{4.38}$$

It remains to calculate the ranks of the subgroups E_k and F_{k+1}. By definition, the rank of E_k coincides with the number of subsets $J \subset \{1, \ldots, N+1\}$ of cardinality $|J| = N + 1 - k$ so that $N + 1 \notin J$ and J is long with respect to ℓ. Examining (4.28) and (4.33), one finds rk $E_k = d_k$. The rank of F_{k+1} is given by the number of subsets $J \subset \{1, \ldots, N+1\}$ so that $|J| = N - k$, $N + 1 \in J$ and J is either short or median with respect to ℓ. Combining (4.27) and (4.33), one concludes that rk $F_{k+1} = c_{N-1-k}$. This completes the proof. $\square$

Chapter 5

Cohomology of Spaces of Polygons

In this chapter we show that the isomorphism type of the graded cohomology ring $H^*(E_d(\ell); \mathbf{Z}_2)$ determines the chamber of the length vector ℓ up to a permutation of the entries of ℓ. This result means that the spaces $E_d(\ell)$ are classified by their $\mathbf{Z}_2$-cohomology rings.

5.1 The inverse Problem

It follows from Proposition 1.5.3 that the homeomorphism type of the space $E_d(\ell)$ depends only on the orbit of the chamber of ℓ under the action of the symmetric group Σ_n. It is an interesting question whether topological invariants of the space $E_d(\ell)$ can be used to distinguish between different orbits.

In [40], K. Walker studied the planar polygon spaces $M_\ell = E_2(\ell)/SO(2)$. He conjectured that these spaces are classified by their cohomology rings, more precisely that if for two generic length vectors ℓ and ℓ' the spaces M_ℓ and $M_{\ell'}$ have isomorphic cohomology rings, then ℓ and ℓ' lie in the same chamber after a permutation of the entries. Walker's Conjecture was proved for a large class of length vectors in [12] and the remaining cases were resolved in [36]. A proof of an analogous result for the spaces $N_\ell = E_3(\ell)/SO(3)$ can also be found in [12].

We will show that for each $d \geq 2$ the spaces $E_d(\ell)$ satisfy an analogue of Walker's Conjecture. We first demonstrate that the Conjecture for the spaces $E_2(\ell)$ and cohomology with integral coefficients follows from the results mentioned above.

By Proposition 1.5.2, for every length vector ℓ there is a homeomorphism

$$E_2(\ell) \simeq S^1 \times M_\ell. \tag{5.1}$$

On the other hand, if ℓ is generic, $0 < \varepsilon < [\ell]$ and (ε, ℓ) denotes the length vector obtained by inserting ε as the first entry of ℓ, then the product $S^1 \times M_\ell$ is homeomorphic to the space $M_{(\varepsilon,\ell)}$:

$$S^1 \times M_\ell \simeq M_{(\varepsilon,\ell)} \tag{5.2}$$

(see Proposition 4.1 in [36], compare also with Proposition 1.6.5). Combining the two homeomorphisms (5.1) and (5.2), we find that if ℓ is generic, then the space $E_2(\ell)$ can be identified as

$$E_2(\ell) \simeq M_{(\varepsilon,\ell)}. \tag{5.3}$$

Suppose now that for two generic length vectors ℓ, ℓ' the spaces $E_2(\ell)$ and $E_2(\ell')$ have isomorphic graded integral cohomology rings. Using the first part of Proposition 1.5.3, we can assume without loss of generality that ℓ and ℓ' are ordered. Fix $0 < \varepsilon < [\ell]$ and $0 < \varepsilon' < [\ell']$ sufficiently small, so that (ε, ℓ) and (ε', ℓ') are ordered as well. Using the homeomorphism (5.3) and Theorem 1.2 from [36], it follows that (ε, ℓ) and (ε', ℓ') lie in the same chamber after a permutation of the entries. Since the length vectors (ε, ℓ) and (ε', ℓ') are ordered, the second part of Lemma 1.3.4 shows that they lie in the same chamber, but then using Lemma 1.3.6 so do the two length vectors ℓ and ℓ'. These arguments establish Walker's Conjecture for the spaces $E_2(\ell)$ as a consequence of the main result of [36].

We can now state the main result of this chapter.

Theorem 5.1.1. *Let ℓ, ℓ' be two generic length vectors and let $d > 2$. The following conditions are equivalent:*

1. *The spaces $E_d(\ell)$ and $E_d(\ell')$ are $O(d)$-equivariantly diffeomorphic;*

2. *There is an isomorphism*

$$H^*(E_d(\ell);\mathbf{Z}_2) \simeq H^*(E_d(\ell');\mathbf{Z}_2)$$

of graded rings;

3. *There is a ring isomorphism*

$$H^{(d-1)*}(E_d(\ell);\mathbf{Z}_2) \simeq H^{(d-1)*}(E_d(\ell');\mathbf{Z}_2);$$

4. *There is a permutation* $\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$, *so that the length vectors* $\sigma(\ell)$ *and* ℓ' *lie in the same chamber.*

The proof of Theorem 5.1.1 uses a similar approach as the results in [12], with the Morse-Bott lacunary principle of Chapter 2 as a central new tool.

Using the results of Chapter 3, it is not difficult to show that the homology groups of the space $E_d(\ell)$ in general do not determine the orbit of the chamber of ℓ under the action of the permutation group Σ_n. For example, consider the two length vectors

$$\ell = (1, 2, 2, 2, 4, 4) \tag{5.4}$$

and

$$\ell' = (1, 1, 3, 4, 8, 8). \tag{5.5}$$

Evidently, ℓ and ℓ' are generic. Moreover, the numbers $a_k(\ell)$ and $a_k(\ell')$ coincide for all k. Thus by Theorem 3.1.1, the $\mathbf{Z}_2$-Betti numbers of spaces $E_d(\ell)$ and $E_d(\ell')$ are equal. Using Theorem 3.1.2, it follows that the integral homology groups of $E_d(\ell)$ and $E_d(\ell')$ are isomorphic if d is even. However, ℓ and ℓ' do not lie in the same chamber since the set $J = \{1, 4, 6\}$ is short with respect to ℓ, but long with respect to ℓ'.

Since the length vectors ℓ and ℓ' given by (5.4) and (5.5) are both ordered, it follows from the second part of Lemma 1.3.4 that there is no permutation $\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$ so that $\sigma(\ell)$ and ℓ' lie in the same chamber. This example shows that to distinguish between different orbits of the Σ_n-action on the set if chambers, it is necessary to use the multiplicative structure of the cohomology ring $H^*(E_d(\ell); \mathbf{Z}_2)$.

The main step in the proof of Theorem 5.1.1 is the computation of the subring $H^{(d-1)*}(E_d(\ell); \mathbf{Z}_2) \subset H^*(E_d(\ell); \mathbf{Z}_2)$, whose result we now state. We again assume for convenience that ℓ is ordered. The general case follows by permuting the entries of ℓ and applying the first part of Proposition 1.5.3.

Proposition 5.1.2. *Let ℓ be an ordered length vector and let $d > 2$. There is an isomorphism of graded rings*

$$H^{(d-1)*}(E_d(\ell); \mathbf{Z}_2) \simeq \Lambda_d(Z_1, \dots, Z_n)/I, \tag{5.6}$$

where $\Lambda_d(Z_1, \dots, Z_n)$ is the exterior algebra over $\mathbf{Z}_2$ on generators $Z_1, \dots, Z_n$ of degree $d-1$ and $I \subset \Lambda_d(Z_1, \dots, Z_n)$ is the ideal generated by all the monomials

$$Z^J = Z_{j_1} \cdots Z_{j_k},$$

so that $J = \{j_1, \dots, j_k\} \subset \{1, \dots, n-1\}$ and the set $J \cup \{n\}$ is long with respect to ℓ.

5.2 Proof of Theorem 5.1.1

We now prove Theorem 5.1.1 assuming Proposition 5.1.2. The proof uses the following algebraic result of J. Gubeladze ([18]).

Given a commutative ring R, an ideal $I \subset R[Z_1, \dots, Z_n]$ is called *monomial* if it is generated by elements of the form $X_1^{a_1} \cdots X_m^{a_m}$, $a_i \geq 0$.

Theorem 5.2.1 ([18])**.** *Let R be a commutative ring and $I \subset R[X_1, \dots, X_m]$, $I' \subset R[Y_1, \dots, Y_{m'}]$ two monomial ideals. Assume that $I \cap \{X_1, \dots, X_m\} = \emptyset$, $I' \cap \{Y_1, \dots, Y_{m'}\} = \emptyset$ and that there is an isomorphism*

$$R[X_1, \dots, X_m]/I \simeq R[Y_1, \dots, Y_{m'}]/I'$$

of R-algebras.

Then $m = m'$ and there is a bijection

$$\{X_1, \ldots, X_m\} \to \{Y_1, \ldots, Y_{m'}\}$$

which maps I to I'.

Proof of Theorem 5.1.1. The implications (1) $\implies$ (2) $\implies$ (3) are evident and the implication (4) $\implies$ (1) follows from the third part of Proposition 1.5.3. It remains to establish the implication (3) $\implies$ (4).

Let ℓ, ℓ' be two generic length vectors so that the rings $H^{(d-1)*}(E_d(\ell); \mathbf{Z}_2)$ and $H^{(d-1)*}(E_d(\ell'); \mathbf{Z}_2)$ are isomorphic and so that, in addition, ℓ and ℓ' are both ordered. We will show that in this case ℓ and ℓ' lie in the same chamber. Since every length vector may be obtained from an ordered length vector by a permutation of the entries, this will establish Theorem 5.1.1.

It follows from Proposition 5.1.2 that $H^{(d-1)*}(E_d(\ell); \mathbf{Z}_2) \simeq \mathbf{Z}_2[Z_1, \ldots, Z_n]/K$ and $H^{(d-1)*}(E_d(\ell'); \mathbf{Z}_2) \simeq \mathbf{Z}_2[Z_1, \ldots, Z_n]/K'$, where K is the ideal generated by the squares Z_j^2, $j = 1, \ldots, n$ and the monomials $Z^J = Z_{j_1} \cdots Z_{j_k}$ so that $n \notin J = \{j_1, \ldots, j_k\}$ and the set $J \cup \{n\}$ is long with respect to ℓ. The monomial ideal K' is defined analogously, but with ℓ replaced by ℓ'.

For $j = 1, \ldots, n-1$, we have $Z_j \in K$ if and only if the set $\{j, n\}$ is long with respect to ℓ. Denote

$$i = max\{1 \leq j \leq n-1 : \ \{j, n\} \text{ is short or median w.r.t. } \ell\}$$

and

$$i' = max\{1 \leq j \leq n-1 : \ \{j, n\} \text{ is short or median w.r.t. } \ell'\}.$$

There is a ring isomorphism

$$H^{(d-1)*}(E_d(\ell); \mathbf{Z}_2) \simeq \mathbf{Z}_2[Z_1, \ldots, Z_i, Z_n]/I, \tag{5.7}$$

where $I \subset \mathbf{Z}_2[Z_1, \ldots, Z_i, Z_n]$ is the ideal generated by the squares Z_j^2, $j \in \{1, \ldots, i, n\}$ as well as the monomials $Z^J = Z_{j_1} \cdots Z_{j_k}$ so that $J = \{j_1, \ldots, j_k\} \subset \{1, \ldots, i\}$ and

$J \cup \{n\}$ is long with respect to ℓ. We note that $Z_j \notin I$ for $j = 1, \ldots, i$. The condition $Z_n \in I$ is equivalent to the one-element set $\{n\}$ being long with respect to ℓ and thus determines the chamber of the length vector ℓ uniquely up to permutation of its entries. Using the first part of Proposition 1.5.1, this case is uniquely characterized by the space $E_d(\ell)$ being empty. Thus we may assume that $Z_n \notin I$.

Similarly,

$$H^{(d-1)*}(E_d(\ell'); \mathbf{Z}_2) \simeq \mathbf{Z}_2[Z_1, \ldots, Z_{i'}, Z_n]/I' \tag{5.8}$$

with the monomial ideal I' defined as above, but with ℓ replaced by ℓ'.

By (5.7) and (5.8), condition (3) of the theorem implies the existence of a ring isomorphism

$$\mathbf{Z}_2[Z_1, \ldots, Z_i, Z_n]/I \simeq \mathbf{Z}_2[Z_1, \ldots, Z_{i'}, Z_n]/I'. \tag{5.9}$$

Applying Theorem 5.2.1, one concludes from (5.9) that $i = i'$ and that there exists a permutation σ of $\{1, \ldots, i, n\}$ so that a subset $J \subset \{1, \ldots, i, n\}$ with $n \in J$ is long with respect to ℓ if and only if the set $\sigma(J)$ is long with respect to ℓ'.

We extend σ to a permutation of $\{1, \ldots, n\}$ by defining

$$\sigma(j) = j \text{ for } j = i+1, \ldots, n-1.$$

Denote by k the image $k = \sigma(n)$ and by α the transposition of the two indices k and n. Let $\alpha(\ell)$ be the length vector obtained from ℓ by interchanging the entries with these indices.

Since for every $j \in \{i+1, \ldots, n-1\}$ the set $\{j, n\}$ is long with respect to both ℓ and ℓ', the composition $\alpha \circ \sigma$ maps the two sets

$$\mathcal{L}_n(\ell) = \{J \subset \{1, \ldots, n\} : n \in J \text{ and } J \text{ is long with respect to } \ell\}$$

and

$$\mathcal{L}_n(\alpha(\ell)) = \{J \subset \{1, \ldots, n\} : n \in J \text{ and } J \text{ is long with respect to } \alpha(\ell)\}$$

bijectively to each other. Since $(\alpha \circ \sigma)(n) = n$, applying the criterion of Corollary 1.3.3, we conclude that the length vectors ℓ and $\alpha(\ell')$ lie in the same chamber after a permutation of their entries. It follows that ℓ and ℓ' also lie in the same chamber after a permutation of the entries. This completes the proof of Theorem 5.1.1. □

5.3 Proof of Proposition 5.1.2

Proof of Proposition 5.1.2. We use notation from the proof of Theorem 3.1.1. Consider the cohomological exact sequence of the pair $(W, E_d(\ell))$:

$$\cdots \to H^{(d-1)k}(W; \mathbf{Z}_2) \xrightarrow{i^k} H^{(d-1)k}(E_d(\ell); \mathbf{Z}_2) \to H^{(d-1)k+1}(W, E_d(\ell); \mathbf{Z}_2) \to \dots$$

By the proof of Theorem 3.1.1, the non-vanishing homology groups of $W - E_d(\ell)$ are concentrated in dimensions which are multiples of $d-1$. Using Poincaré duality and excision,

$$H^{(d-1)k+1}(W, E_d(\ell); \mathbf{Z}_2) \simeq H_{(d-1)(n-k)-1}(W - E_d(\ell); \mathbf{Z}_2) = 0.$$

Thus the homomorphism

$$i^k : H^{(d-1)k}(W; \mathbf{Z}_2) \to H^{(d-1)k}(E_d(\ell); \mathbf{Z}_2)$$

induced by inclusion is surjective. On the other hand, from the exact sequence

$$H^{(d-1)k}(W, E_d(\ell); \mathbf{Z}_2) \xrightarrow{h^k} H^{(d-1)k}(W; \mathbf{Z}_2) \xrightarrow{i^k} H^{(d-1)k}(E_d(\ell); \mathbf{Z}_2)$$

and the commutative square

$$\begin{tikzcd}
H_{(d-1)(n-k)}(W - E_d(\ell); \mathbf{Z}_2) \arrow[r, "j_{n-k}"] \arrow[d, "PD"] & H_{(d-1)(n-k)}(W; \mathbf{Z}_2) \arrow[d, "PD"] \\
H^{(d-1)k}(W, E_d(\ell); \mathbf{Z}_2) \arrow[r, "h^k"] & H^{(d-1)k}(W; \mathbf{Z}_2),
\end{tikzcd}$$

where the columns are Poincaré duality maps, we see that the kernel of i^k consists exactly of the Poincaré duals of the elements of the image of j_{n-k}.

For $j = 1, \dots, n$, denote by

$$\pi_j : (S^{d-1})^n \to S^{d-1}$$

the projection to the jth factor. We identify $H^*(W;\mathbf{Z}_2)$ as the exterior algebra generated by the classes

$$X_j = \pi_j^*[S^{d-1}] \in H^{d-1}(E_d(\ell);\mathbf{Z}_2),$$

where $[S^{d-1}] \in H^{d-1}(S^{d-1};\mathbf{Z}_2)$ is the generator. Then

$$H^{(d-1)*}(E_d(\ell);\mathbf{Z}_2) \simeq \Lambda(X_1,\ldots,X_n)/I$$

for $I = \ker(i^k)$.

Using Corollary 3.5.6, the kernel of i^k consists of the Poincaré duals of all the classes $[V_J],[W_J] \in H_{(d-1)*}(W;\mathbf{Z}_2)$ so that $J \subset \{1,\ldots,n\}$ is a long subset containing the index n. The Poincaré dual of the class $[V_J]$ is the product $X^J = X_{j_1}\cdots X_{j_k}$, $J = \{j_1,\ldots,j_k\}$. Next, we note that

$$[W_J] = \sum_{j\in J}[V_{J-\{j\}}]. \tag{5.10}$$

Indeed, denoting for $j = 1,\ldots,n$ by $S_j^{d-1} \subset W$ the jth factor of the product $W = (S^{d-1})^n$, we see that W_J may be identified with the product of the diagonal $\Delta \subset \Pi_{j\in J}S_j^{d-1}$ and $\Pi_{j\notin J}S_j^{d-1}$. Equation (5.10) now follows from the fact that the homology class of the diagonal in a product $S^{d-1}\times\cdots\times S^{d-1}$ is the sum of the homology classes of the factors.

It follows that the Poincaré dual of the class $[W_J]$ is

$$PD([W_J]) = \sum_{j\in J} X^{J-\{j\}}.$$

We conclude that the kernel of i^k is generated additively by all the monomials

$$X^J = X_{j_1}\cdots X_{j_k}$$

as well as the polynomials

$$\sum_{j\in J} X^{J-\{j\}},$$

so that the subset $J = \{j_1,\ldots,j_k\} \subset \{1,\ldots,n\}$ is long with respect to ℓ and contains the index n.

Let us consider the following change of variables: we define a basis $Z_1, \ldots, Z_n$ of $H^*(W; \mathbf{Z}_2)$ by

$$Z_j = X_j + X_n,\ j = 1, \ldots, n-1$$

and

$$Z_n = X_n.$$

Then for $J = \{j_1, \ldots, j_k\} \subset \{1, \ldots, n\}$ the monomial $Z^J = Z_{j_1} \cdots Z_{j_k}$ can be expressed as

$$Z^J = \begin{cases} X^J & \text{if } n \in J, \\ X^J + \sum_{j \in J} X^{J - \{j\}} X_n & \text{if } n \notin J. \end{cases}$$

Thus if we denote for a subset $J \subset \{1, \ldots, n\}$ by J' the set $J - \{n\}$, then

$$Z^{J'} = \sum_{j \in J} X^{J - \{j\}}.$$

We see that in this new basis $I \subset \Lambda_d(Z_1, \ldots, Z_n)$ is additively generated by all the monomials Z^J so that either $n \in J$ and J is long with respect to ℓ, or $n \notin J$ and the set $J \cup \{n\}$ is long with respect to ℓ. Thus a multiplicative basis for I is given by all the monomials Z^J so that $n \notin J$ and the set $J \cup \{n\}$ is long with respect to ℓ. This completes the proof of Proposition 5.1.2. $\square$

Chapter 6

Conclusions

The study of polygon spaces is an exciting field where algebraic topology, Morse theory and combinatorics intertwine. In this thesis, we showed that the special properties of the robot arm distance map in higher dimensions lead to new insights in Morse-Bott theory as well as to results on the structure of the homology groups and of the cohomology rings of spaces $E_d(\ell)$ of polygons up to translations.

The results of Chapter 3 together with the previous work of M. Farber and D. Schuetz ([15]) give explicit expressions for the homology groups $H_*(E_d(\ell); \mathbf{Z})$ in the case where d is even. The Morse-Bott lacunary principle of Chapter 2 also provides a method for computing the integral homology groups when d is odd, however in this case it seems a much more difficult task to find explicit general expressions.

As an example, let us list the integral homology groups in the case where ℓ is generic and $n = 5$. There are seven chambers (up to permutation of the entries of ℓ) and the homology groups $H_*(E_d(\ell); \mathbf{Z})$ for d odd are given below. (Only the groups $H_p(E_d(\ell); \mathbf{Z})$ for $p = d - 2$, $p = d - 1$ and $p = 2d - 3$ are listed. The remaining groups $H_p(E_d(\ell); \mathbf{Z})$ can be recovered from the table using Theorem 3.1.2, Poincaré duality and the universal coefficient Theorem.)

ℓ	$H_{d-2}(E_d(\ell);\mathbf{Z})$	$H_{d-1}(E_d(\ell);\mathbf{Z})$	$H_{2d-3}(E_d(\ell);\mathbf{Z})$
$(1,1,1,1,5)$	0	0	0
$(1,1,3,3,3)$	$\mathbf{Z}_2$	$\mathbf{Z}\oplus\mathbf{Z}$	$\mathbf{Z}_2\oplus\mathbf{Z}_2\oplus\mathbf{Z}$
$(1,1,1,1,3)$	0	$\mathbf{Z}$	0
$(1,2,2,2,4)$	0	$\mathbf{Z}\oplus\mathbf{Z}$	$\mathbf{Z}$
$(1,1,2,2,3)$	0	$\bigoplus_3 \mathbf{Z}$	$\mathbf{Z}_2\oplus\mathbf{Z}_2$
$(1,1,1,2,2)$	0	$\bigoplus_4 \mathbf{Z}$	$\mathbf{Z}_2\oplus\mathbf{Z}_2\oplus\mathbf{Z}_4$
$(1,1,1,1,1)$	0	$\bigoplus_5 \mathbf{Z}$	$(\bigoplus_4 \mathbf{Z}_2)\oplus\mathbf{Z}_3$

It is an interesting problem for further research to study in detail the structure of the homology groups in the case where d is odd. A first step in this direction is the torsion criterion of Theorem 3.1.5.

In Chapter 4, we considered configuration spaces of planar polygons in the case where one of the edges has variable length. We computed their homology groups and related our results to topological questions studied in thermodynamics. Interpreting spaces of telescopic polygons as energy sublevels of the anti-ferromagnetic mean-field XY model allows to obtain an explicit formulae for the Betti numbers of the sublevels. As an indicator of the presence of phase transitions, we studied the exponential growth rate of the total Betti number as opposed to the exponential growth rate of the Euler characteristic, which was considered in previous results. One may hope that using the total Betti number instead of the Euler characteristic provides a more sensitive tool for the study of different versions of the Topological Hypothesis.

In Chapter 5, we studied the inverse problem for the spaces $E_d(\ell)$ of polygons up to translations. We proved that the spaces are classified by their $\mathbf{Z}_2$-cohomology rings. This motivates further study of the spaces $E_d(\ell)$ as analogues in higher dimensions of M_ℓ and N_ℓ.

Appendices

Appendix A

Spaces of polygonal Chains

In the main body of this thesis, the spaces $E_d(\ell)$ of polygons in Euclidean space $\mathbb{R}^d$ were studied. Denote by $p : E_d(\ell) \to S^{d-1}$ the restriction to $E_d(\ell) \subset W = (S^{d-1})^n$ of the projection $(u_1, \ldots, u_n) \mapsto u_n$. For $e \in S^{d-1}$, the preimage $C_d(\ell) = p^{-1}(e)$ may be viewed as the space of polygons up to translations, where the direction of the edge l_n is fixed. The spaces $C_d(\ell)$ were studied in [20], [21] and in [13].

In the current appendix, we gather basic relationships between the spaces $E_d(\ell)$ and $C_d(\ell)$. We also show that for $d \neq 3$, the graded isomorphism type of the $\mathbf{Z}_2$-cohomology ring of the space $C_d(\ell)$ is invariant under arbitrary permutations of the entries of ℓ.

A.1 Spaces of polygonal Chains

We recall from Section 1 the notation $W = (S^{d-1})^n$. Fix $e \in S^{d-1}$ and consider the subspace

$$W' = \{(u_1, \ldots, u_n) \in W : u_n = -e\} \subset W.$$

One defines $C_d(\ell) \subset W'$ as the intersection $C_d(\ell) = E_d(\ell) \cap W'$. There is an action on $C_d(\ell)$ of the subgroup $O(d-1) \subset O(d)$ consisting of those elements of the orthogonal group $O(d)$, which fix the vector e.

The following properties of the spaces $C_d(\ell)$ were established in [20] and [21].

Proposition A.1.1. *([20],[21])*

1. *For every generic length vector ℓ, the space $C_d(\ell)$ is a closed oriented manifold of dimension*
$$dim\ C_d(\ell) = (d-1)(n-2) - 1.$$

2. *If ℓ and ℓ' are generic and $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ is a permutation which fixes the index n, then the spaces $C_d(\ell)$ and $C_d(\sigma(\ell))$ are $O(d-1)$-equivariantly diffeomorphic.*

3. *Let ℓ be a generic length vector and let $0 < \varepsilon < [\ell]$ (see Section 1.3 for the definition of the quantity $[\ell]$). Then there is a diffeomorphism*
$$C_d(\varepsilon, \ell) \simeq S^{d-1} \times C_d(\ell).$$
Here (ε, ℓ) is the length vector obtained from ℓ by inserting ε as the first entry.

The following proposition states basic relationships between the spaces $E_d(\ell)$ and $C_d(\ell)$.

Proposition A.1.2. 1. *For every length vector ℓ there is a fibration*

$$\begin{tikzcd} C_d(\ell) \arrow[r] & E_d(\ell) \arrow[d, "p"] \\ & S^{d-1}. \end{tikzcd}$$

2. *Let ℓ be a generic length vector and let $0 < \varepsilon < [\ell]$. There is a diffeomorphism*
$$E_d(\ell) \simeq C_d(\ell, \varepsilon).$$
Here (ℓ, ε) denotes the length vector obtained from ℓ by inserting ε as the last entry.

Proof. To prove the first part of the proposition, let $e \in S^{d-1}$ and consider an open neighbourhood $e \in U \subset S^{d-1}$ and a section $\psi : U \to SO(d)$ of the bundle

$SO(d) \to S^{d-1}$. We define a local trivialization of $p : E_d(\ell) \to S^{d-1}$ by mapping $(e', u) \in U \times C_d(\ell)$ to

$$(\psi(e')u_1, \ldots, \psi(e')u_{n-1}, -\psi(e')e) \in E_d(\ell) \subset W,$$

where $u = (u_1, \ldots, u_{n-1}, -e) \in C_d(\ell) \subset W'$. This defines the structure of a locally trivial fibre bundle.

To prove the second assertion, consider the map $G : (\mathbb{R}^d)^n \to \mathbb{R}^n \times \mathbb{R}^d$,

$$G(v_1, \ldots, v_n) = (|v_1|, |v_2 - v_1|, \ldots, |v_n - v_{n-1}|, v_n).$$

The restriction of G to the subset

$$Z = \{(v_1, \ldots, v_n) \in (\mathbb{R}^d)^{n-1} : v_1 \neq 0 \text{ and } v_{j+1} \neq v_j \text{ for } j = 1, \ldots, n-1\}$$

is smooth. One concludes analogously as in the Proof of Lemma 1.5.4 that every element $(l_1, \ldots, l_n, v_n) \in \mathbb{R}^{n-1}_{>0} \times \mathbb{R}^d$, so that the length vector $(l_1, \ldots, l_n, |v_{n-1}|)$ is generic, is a regular value of $G|_Z$. Moreover, there are diffeomorphisms

$$G^{-1}(\ell, \varepsilon e) \simeq C_d(\ell, \varepsilon)$$

and

$$G^{-1}(\ell, 0) \simeq E_d(\ell).$$

If $0 < \varepsilon < [\ell]$, then G has no critical values on the interval in $\mathbb{R}^n \times \mathbb{R}^d$ connecting the two points $(\ell, 0)$ and $(\ell, \varepsilon e)$. Thus in this case there are diffeomorphisms

$$C_d(\ell, \varepsilon) \simeq G^{-1}(\ell, \varepsilon e) \simeq G^{-1}(\ell, 0) \simeq E_d(\ell).$$

$\square$

A.2 Cohomology of Spaces of polygonal Chains

In light of the second part of Proposition A.1.1, a possible analogue of Walker's Conjecture for the spaces $C_d(\ell)$ is the question whether for generic ℓ the graded isomorphism type of the cohomology ring $H^*(C_d(\ell); \mathbf{Z}_2)$ determines the chamber of

ℓ up to a permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ with $\sigma(n) = n$. In [13], it was shown that the answer to this question is positive if one only considers length vectors that are *dominated*, i.e. so that l_n is a maximal entry of ℓ. We now show that if $d > 3$, then the answer is negative in general, both for cohomology with coefficients in $\mathbf{Z}_2$ as well as for integral cohomology.

Let $n > 3$. Suppose that ℓ is an ordered length vector with $a_{n-3}(\ell) = 1$ (see Proposition 1.6.6) and that ℓ' is obtained from ℓ by interchanging the entries l_{n-3} and l_n. Explicitly, ℓ and ℓ' can be chosen as follows. Let $0 < \varepsilon < 1/(n-3)$. One defines ℓ and ℓ' to be the length vectors with entries

$$l_j = \varepsilon \text{ for } j \notin \{n-2, n-1, n\},\ l_{n-2} = l_{n-1} = l_n = 1 \tag{A.1}$$

and

$$l'_j = \varepsilon \text{ for } j \notin \{n-3, n-2, n-1\},\ l'_{n-3} = l'_{n-2} = l'_{n-1} = 1. \tag{A.2}$$

Note that ℓ is dominated while ℓ' is not.

Proposition A.2.1. *Let $n > 3$ and let ℓ and ℓ' be the length vectors given by (A.1) and (A.2). There are diffeomorphisms*

$$C_d(\ell) \simeq (S^{d-1})^{n-3} \times S^{d-2}$$

and

$$C_d(\ell') \simeq (S^{d-1})^{n-4} \times T^1 S^{d-1}.$$

Proof. The first diffeomorphism follows from $C_d(1,1,1) \simeq S^{d-2}$ by successive application of the third part of Proposition A.1.1. The second diffeomorphism follows by combining the second part of Proposition A.1.2 with Proposition 1.6.6. □

Corollary A.2.2. *Let $d > 3$. If ℓ and ℓ' are as in (A.1) and (A.2), then the graded cohomology rings $H^*(C_d(\ell); \mathbf{Z}_2)$ and $H^*(C_d(\ell'); \mathbf{Z}_2)$ are isomorphic. If, in addition, d is even, then the graded cohomology rings $H^*(C_d(\ell); \mathbf{Z})$ and $H^*(C_d(\ell'); \mathbf{Z})$ are isomorphic as well. However, there is no permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ so that $\sigma(n) = n$ and the length vectors $\sigma(\ell)$ and ℓ' lie in the same chamber.*

Proof. In the case $d > 3$ the ring $H^*(T^1S^{d-1}; \mathbf{Z}_2)$ is uniquely determined by Poincaré duality and the fact that the non-vanishing groups $H^p(T^1S^{d-1}; \mathbf{Z}_2)$ lie in dimensions $p = 0, p = d-2, p = d-1$ and $p = 2(d-1)-1$ and are all isomorphic to $\mathbf{Z}_2$ (see Example 3.2.5). The first assertion follows using Proposition A.2.1 and the Künneth theorem for cohomology. The second assertion follows analogously. The last claim of the Corollary follows from the fact that ℓ admits a long two-element subset $J \subset \{1, \ldots, n\}$ with $n \in J$ while ℓ' does not. $\square$

In [13], the subring $H^{(d-1)*}(C_d(\ell); \mathbf{Z}_2) \subset H^*(C_d(\ell); \mathbf{Z}_2)$ was computed in the case where the length vector ℓ is dominated. One can use similar arguments as were employed in the proof of Theorem 5.1.1 to extend the computation to the general case:

Proposition A.2.3. *Let ℓ be a (not necessarily dominated) length vector and let $d > 2$. Let $m \in \{1, \ldots, n\}$ be the index of any maximal entry of ℓ. There is an isomorphism of graded rings*

$$H^{(d-1)*}(C_d(\ell); \mathbf{Z}_2) \simeq \Lambda_d(Z_1, \ldots, Z_{\hat{m}}, \ldots Z_n)/I,$$

where $\Lambda_d(Z_1, \ldots, Z_{\hat{m}}, \ldots Z_n)$ is the exterior algebra on generators $Z_1, \ldots, Z_n$ of degree $d-1$ and I is the ideal generated by all the monomials $Z^J = Z_{j_1} \cdots Z_{j_k}$ so that $J = \{j_1, \ldots, j_k\} \subset \{1, \ldots, \hat{m}, \ldots, n\}$ and the set $J \cup \{m\}$ is long with respect to ℓ.

In the case where ℓ is dominated, $m = n$ and Proposition A.2.3 recovers the result of the computation in Section 3 of [13].

If $d > 3$ and ℓ is generic, then the cohomology ring $H^*(C_d(\ell); \mathbf{Z}_2)$ can be recovered from the result of Proposition A.2.3 and Poincaré duality:

Proposition A.2.4. *Let ℓ be a generic length vector and let $m \in \{1, \ldots, n\}$ be the index of any maximal entry of ℓ. Assume that $d > 3$. The cohomology $H^*(C_d(\ell); \mathbf{Z}_2)$ is additively generated by classes*

$$Z^J \in H^{(d-1)|J|}(C_d(\ell); \mathbf{Z}_2),\ Y^K \in H^{(d-1)(n-2-|K|)-1}(C_d(\ell); \mathbf{Z}_2),$$

so that $J, K \subset \{1, \ldots, \hat{m}, \ldots, n\}$ *and the sets* $J \cup \{m\}$ *and* $K \cup \{m\}$ *are short with respect to* ℓ*. The product structure is given by*

$$Z^J Z^K = \begin{cases} Z^{J \cup K} & \textit{if } J \cap K = \emptyset \textit{ and } J \cup K \cup \{m\} \textit{ is short w.r.t. } \ell, \\ 0 & \textit{if } J \cap K \neq \emptyset \textit{ or } J \cup K \cup \{m\} \textit{ is long w.r.t. } \ell, \end{cases}$$

$$Z^J Y^K = \begin{cases} Y^{K-J} & \textit{if } J \subset K, \\ 0 & \textit{if } J \not\subseteq K \end{cases}$$

and

$$Y^J Y^K = 0 \textit{ for all } J, K.$$

Proof of Proposition A.2.4. The non-vanishing groups $H^p(C_d(\ell); \mathbf{Z}_2)$ lie in dimensions

$$p = (d-1)k,\ 0 \leq k \leq n-3 \text{ and } p = (d-1)k - 1,\ 1 \leq k \leq n-2.$$

By Proposition A.2.3, an additive basis of the group $H^{(d-1)k}(C_d(\ell); \mathbf{Z}_2)$ is given by classes Z^J so that $J \subset \{1, \ldots, \hat{m}, \ldots, n\}$, $|J| = k$ and the set $J \cup \{m\}$ is short with respect to ℓ. There are isomorphisms

$$H^{(d-1)(n-2-|J|)-1}(C_d(\ell); \mathbf{Z}_2) \simeq H_{(d-1)|J|}(C_d(\ell); \mathbf{Z}_2) \tag{A.3}$$

$$\simeq \mathrm{Hom}(H^{(d-1)|J|}(C_d(\ell); \mathbf{Z}_2), \mathbf{Z}_2), \tag{A.4}$$

where the first isomorphism is given by Poincaré duality and the second isomorphism follows from the universal coefficient theorem. We define

$$Y^J \in H^{(d-1)(n-2-|J|)-1}(C_d(\ell); \mathbf{Z}_2)$$

to be the elements of a dual basis under the identifications of (A.3) and (A.4) to the basis of $H^{(d-1)|J|}(C_d(\ell); \mathbf{Z}_2)$ given by the classes Z^J. Thus the classes Y^J, where $|J| = k$, $m \notin J$ and $J \cup \{m\}$ is short with respect to ℓ, form a basis of $H^{(d-1)(n-2-k)-1}(C_d(\ell); \mathbf{Z}_2)$. In particular, $Y^{\emptyset}$ is the generator of the group $H^{(d-1)(n-2)-1}(C_d(\ell); \mathbf{Z}_2)$.

It remains to show that products of classes Z^J, Y^K can be expressed as in the claim of the proposition. In the case of products of the form $Z^J Z^K$, this follows from Proposition A.2.3. Since for all I, J, K with $|I| + |J| = |K|$,

$$Z^I(Z^J Y^K) = Z^{I \cup J} Y^K = \begin{cases} [C_d(l)] & \text{if } I \cup J = K, \\ 0 & \text{if } I \cup J \neq K, \end{cases} \tag{A.5}$$

we find that the product $Z^J Y^K$ vanishes if $J \not\subseteq K$. In the case where $J \subset K$, it follows from (A.5) and the definition of the classes Y^J that $Z^J Y^K = Y^{K-J}$. Finally, if $d > 3$, then products of the form $Y^J Y^K$ vanish for dimensional reasons. □

Using the diffeomorphism $E_d(\ell) \simeq C_d(\ell, \varepsilon)$ established in Proposition A.1.2, one obtains as a special case of Proposition A.2.4 a computation of the cohomology ring $H^*(E_d(\ell); \mathbf{Z}_2)$ for $d > 3$ and generic ℓ.

In the case $d = 2$, $C_d(\ell)$ coincides with the space M_ℓ of planar polygons, viewed up to all orientation-preserving Euclidean isometries. In particular, the homeomorphism type of the space $C_2(\ell)$ is invariant under arbitrary permutations of the entries of ℓ. Together with the explicit description given in Proposition A.2.4 of the cohomology ring $H^*(C_d(\ell); \mathbf{Z}_2)$ for $d > 3$, one concludes:

Corollary A.2.5. *Let ℓ be a generic length vector. If $d \neq 3$, then for every permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ the graded cohomology rings $H^*(C_d(\ell); \mathbf{Z}_2)$ and $H^*(C_d(\sigma(\ell)); \mathbf{Z}_2)$ are isomorphic. Here $\sigma(\ell)$ denotes the length vector obtained from ℓ by permuting the entries by σ.*

Note that the permutation σ in Corollary A.2.5 may not fix the index n. Recall that $C_d(\ell)$ is the space of polygons up to translations, so that the direction of one of the edges in d-space is fixed. The Corollary means that in the case of $d \neq 3$, the isomorphism type of the ring $H^*(C_d(\ell); \mathbf{Z}_2)$ is independent of the choice of the edge whose direction is fixed.

A.3 Proof of Proposition A.2.3

As noted above, it suffices to consider the case where the length vector ℓ is non-dominated, i.e. when a maximal entry of ℓ has index $m \in \{1, \dots, n-1\}$.

We recall that the space $C_d(\ell)$ is the intersection of $E_d(\ell) \subset W = (S^{d-1})^n$ with the subspace $W' = (S^{d-1})^{n-1} \times \{-e\} \subset W$. Denote as in Section 3.5 for a subset $J \subset \{1, \dots, n\}$ by $W_J \subset W$ the submanifold

$$W_J = \{(u_1, \dots, u_n) \in W : \; u_i = u_j \text{ for } i, j \in J\}.$$

Let W'_J be the intersection $W'_J = W_J \cap W'$. We note that if $n \in J$, then the submanifold W'_J is given by

$$\{(u_1, \dots, u_{n-1}) \in (S^{d-1})^{n-1} : u_j = -e \text{ for } j \in J\}. \tag{A.6}$$

On the other hand, if $n \notin J$, then W'_J is given by

$$\{(u_1, \dots, u_{n-1}) \in (S^{d-1})^{n-1} : u_i = u_j \text{ for } i, j \in J\}. \tag{A.7}$$

Comparing (A.6) and (A.7) with equations (3.10) and (3.11) in Section 3.5 and applying Lemma 3.5.3, we find that a basis of $H_*(W'; \mathbf{Z}_2)$ is given by the classes $[W'_J]$, so that $J \subset \{1, \dots, n\}$ is a subset with $m \in J$ (we recall that $m \in \{1, \dots, n-1\}$ is the index of a maximal entry of ℓ).

Lemma A.3.1. *Let ℓ be a (not necessarily dominated) length vector. The image of the homomorphism*

$$j_k : H_{(d-1)k}(W' - C_d(\ell); \mathbf{Z}_2) \to H_{(d-1)k}(W'; \mathbf{Z}_2)$$

induced by inclusion is generated by all the classes $[W'_J]$, so that $J \subset \{1, \dots, n\}$ is a subset of cardinality $|J| = n - k$ which is long with respect to ℓ and contains the index m.

We now prove Proposition A.2.4 assuming Lemma A.3.1. For $j = 1, \dots, n-1$, denote by $\pi_j : W' \to S^{d-1}$ the projection to the jth factor. Let $X_j \in H^{d-1}(W'; \mathbf{Z}_2)$ be the pull-back $X_j = \pi_j^*([S^{d-1}])$ of the generator $[S^{d-1}] \in H^{d-1}(S^{d-1}; \mathbf{Z}_2)$. It follows from

(A.6) and (A.7) that the Poincaré dual of the class $[W_J] \in H_{(d-1)(n-|J|)}(W'; \mathbf{Z}_2)$ may be expressed as follows:

$$PD([W_J]) = \begin{cases} X^{J-\{n\}} & \text{if } n \in J, \\ \sum\limits_{j\in J} X^{J-j} & \text{if } n \notin J. \end{cases} \tag{A.8}$$

Here for as subset $J = \{j_1, \ldots, j_k\} \subset \{1, \ldots, n\}$, X^J denotes the monomial $X^J = X_{j_1} \cdots X_{j_k}$.

Arguing as in the proof of Proposition 5.1.2 in Section 5.3, the homomorphism $H^{(d-1)*}(C_d(\ell); \mathbf{Z}_2) \to H^{(d-1)*}(W'; \mathbf{Z}_2)$ induced by inclusion is surjective and its kernel $I \subset H^{(d-1)*}(C_d(\ell); \mathbf{Z}_2)$ is the ideal generated by the Poincaré duals of the elements of the image of $j_* : H_{(d-1)*}(W' - C_d(\ell); \mathbf{Z}_2) \to H_{(d-1)*}(W'; \mathbf{Z}_2)$. Using (A.8), it follows that

$$H^{(d-1)*}(C_d(\ell); \mathbf{Z}_2) \simeq \Lambda_d(X_1, \ldots, X_{n-1})/I,$$

where $I \subset \Lambda_d(X_1, \ldots, X_{n-1})$ is the ideal generated by all the monomials X^K, so that $K \subset \{1, \ldots, n-1\}$, $m \in K$ and the set $K \cup \{n\}$ is long with respect to ℓ as well as all the sums $\sum_{j\in J} X^{J-j}$, so that $J \subset \{1, \ldots, n-1\}$, $m \in J$ and J is long with respect to ℓ.

Consider the following change of variables: we denote

$$Z_j = X_j + X_m \text{ for } j \neq m, n \text{ and } Z_n = X_m.$$

Then for every subset $J \subset \{1, \ldots, n-1\}$ with $m \in J$,

$$X^J = Z^{J-\{m\}\cup\{n\}} \text{ and } \sum_{j\in J} X^{J-\{j\}} = Z^{J-\{m\}}.$$

It follows that I corresponds to the ideal in $\Lambda_d(Z_1, \ldots, Z_{\hat{m}}, \ldots, Z_n)$ generated by all the monomials Z^J, so that $J \subset \{1, \ldots, \hat{m}, \ldots n\}$ and the set $J \cup \{m\}$ is long with respect to ℓ. This completes the proof of Proposition A.2.4.

The proof of Lemma A.3.1 uses the explicit description of the homology $H_*(W' - C_d(\ell); \mathbf{Z}_2)$ which was obtained in [13]. Namely, it was shown in Lemma 1.3 of [13] that the homology $H_*(W' - C_d(\ell); \mathbf{Z}_2)$ of the complement of $C_d(\ell)$ in W' has a basis consisting of all the classes $[W'_J]$, so that the subset $J \subset \{1, \dots, n\}$ is long with respect to ℓ. Similarly as in the proof of Lemma 3.5.5 in Section 3.5, we consider direct sum decompositions

$$H_{(d-1)k}(W' - C_d(\ell); \mathbf{Z}_2) = A_k \oplus A'_k \oplus B_k \oplus B'_k$$

and

$$H_{(d-1)k}(W'; \mathbf{Z}_2) = A_k \oplus B_k \oplus C_k \oplus D_k,$$

where

- A_k (respectively A'_k) is generated by the classes $[W'_J]$ so that $|J| = n - k$, $n \notin J$, J is long with respect to ℓ and $m \in J$ (respectively $m \notin J$).
- B_k (respectively B'_k) is generated by the classes $[W'_J]$ so that $|J| = n - k$, $n \in J$, J is long with respect to ℓ and $m \in J$ (respectively $m \notin J$).

Moreover,

- C_k is generated by the classes $[W'_J]$ so that $|J| = n - k$, $m \in J$, J is short or median with respect to ℓ and $n \notin J$.
- D_k is generated by the classes $[W'_J]$ so that $|J| = n - k$, $m \in J$, J is short or median with respect to ℓ and $n \in J$.

Using the formula

$$[W'_J] \cdot [W'_K] = \begin{cases} 1 & \text{if } |J \cap K| = 1, \\ 0 & \text{if } |J \cap K| > 1 \end{cases}$$

for the intersection number of any two classes $[W'_J], [W'_K]$ with $|J| + |K| = n + 1$ and repeating the arguments in the proof of Lemma 3.5.5, one concludes that the image of the inclusion homomorphism j_k is given by $A_k \oplus B_k$. This completes the proof of Lemma A.3.1.

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